

# ON ZIPPIN'S EMBEDDING THEOREM OF BANACH SPACES INTO BANACH SPACES WITH BASES

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**ABSTRACT.** We present a new proof of Zippin's Embedding Theorem, that every separable reflexive Banach space embeds into one with shrinking and boundedly complete basis, and every Banach space with a separable dual embeds into one with a shrinking basis. This new proof leads to improved versions of other embedding results.

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## 1. INTRODUCTION

In 1988 M. Zippin answered a question posed by Pelczyński [25, Problem I] in 1964 and proved the following embedding result.

**Theorem 1.1.** [29, Corollary] *Every separable and reflexive Banach space embeds into a reflexive Banach space with a basis.*

It was shown in [3], and mentioned in [29], that Theorem 1.1 can be deduced from the following result which answers a question of Lindenstrauss and Tzafriri [19, Problem 1.b.16].

**Theorem 1.2.** [29, Theorem] *Every Banach space with a separable dual embeds into a space with shrinking basis.*

Zippin's Theorem is the starting point of several other embedding results. In [24], it was shown that if  $X$  is a reflexive and separable Banach space and  $\alpha$  is a countable ordinal for which  $\max(\text{Sz}(X), \text{Sz}(X^*)) \leq \omega^{\alpha\omega}$  then  $X$  embeds into a reflexive space  $Z$  with basis for which  $\max(\text{Sz}(Z), \text{Sz}(Z^*)) \leq \omega^{\alpha\omega}$ . Here  $\text{Sz}(Y)$  denotes the Szlenk index of a Banach space  $Y$  [28] (see Section 4). In [9] it was shown that if  $X$  has a separable dual and  $\text{Sz}(X) \leq \omega^{\alpha\omega}$ , then  $X$  embeds in a space  $Z$  with shrinking basis for which  $\text{Sz}(Z) \leq \omega^{\alpha\omega}$ . Causey [4, 5] refined these results and proved that if  $\text{Sz}(X) \leq \omega^\alpha$ , then  $X$  embeds into a space  $Z$  with

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2000 *Mathematics Subject Classification.* 46B03, 46B10.

*Key words and phrases.* Embedding into Banach spaces with bases, Szlenk index.

Research partially supported by grants from the National Science Foundation DMS 0856148 and DMS 1160633.

a shrinking basis with  $\text{Sz}(Z) \leq \omega^{\alpha+1}$ , and it embeds into a space  $Z$  with shrinking and boundedly complete basis for which  $\max(\text{Sz}(Z)\text{Sz}(Z^*)) \leq \omega^{\alpha+1}$ , in case that  $X$  is reflexive and  $\max(\text{Sz}(X)\text{Sz}(X^*)) \leq \omega^\alpha$ . Recall that by [1, Theorems 3.22 and 4.2] the Szlenk index of a space with separable dual is always of the form  $\omega^\alpha$ , for some  $\alpha < \omega_1$ . In [16] Johnson and Zheng characterized reflexive spaces, which embed into reflexive spaces, having an unconditional basis, and in [17] they obtained an analogous result for spaces with separable duals.

The proof of all these embedding results start by applying Theorem 1.1 or Theorem 1.2 to embed the given space  $X$  into a reflexive space or a space with separable dual  $Z$ , which has a basis. Then, using the additional properties of  $X$ , one modifies the space  $Z$  appropriately, to achieve the wanted properties of  $Z$ , without losing the embeddability of  $X$  into  $Z$ .

The two known proofs of Zippin's Embedding Theorem 1.2, namely Zippin's original proof, as well as the proof by Ghoussoub, Maurey and Schachermayer [11] start by embedding the given Banach space  $X$  into  $C(\Delta)$ , the space of continuous functions on the Cantor set  $\Delta$ , and then passing to subspaces and modifying the norm on them. Unfortunately, neither proof provides additional information about the space with basis in which  $X$  embeds. In this paper we will follow a different approach and present a proof of Theorems 1.1 and 1.2 which starts from a Markushevich basis of the given space  $X$ , and then extends and modifies this Markushevich basis just enough to arrive to a space with shrinking basis. The resulting space  $W$  will then be much closer to the space  $X$  and inherit several properties.

Our main result is as follows. All possibly unfamiliar notation will be introduced later.

**Main Theorem.** *Assume that  $X$  is a Banach space with separable dual. Then  $X$  embeds into a space  $W$  with a shrinking basis  $(w_i)$  so that*

- a)  $\text{Sz}(W) = \text{Sz}(X)$ ,
- b) *if  $X$  is reflexive then  $W$  is reflexive and  $\text{Sz}(X^*) = \text{Sz}(W^*)$ , and*
- c) *if  $X$  has the  $w^*$ -Unconditional Tree Property, then  $(w_i)$  is unconditional.*

Part (a) and (b) of the Main Theorem answer a question posed by Pelczyński, and sharpen the results of [4, 5, 9, 24] which were stated at the beginning of this section. As mentioned before, the fact that a reflexive separable Banach space  $X$ , or a space with separable dual, having the  $w^*$ -Unconditional Tree Property (see Section 6) embeds into a space  $Y$  with boundedly and shrinking basis, or with shrinking basis, respectively, was first shown in [16] and [17]. While the proofs of the main results of [16] start out by using Zippin's Theorem 1.1 and first consider an embedding of the given separable space  $X$  into a reflexive space with basis, in [17] it was directly shown that a space with separable dual and the  $w^*$ -Unconditional Tree Property, embeds into one with an unconditional and shrinking basis. Our argument will follow more along the lines of [17] and use coordinate systems which are known to exist in every separable Banach space, namely Markushevich bases, and their multidimensional counterparts *Finite Dimensional Markushevich Decompositions* (see Section 2).

The Main Theorem will follow from the following two results, Theorem A and Theorem B. The first one is a version of the Main Theorem for *Finite Dimensional Decompositions* (FDD), which will be defined in Section 2.

**Theorem A.** *Assume that  $X$  is Banach space with separable dual. Then  $X$  embeds into a space  $Z$  with a shrinking FDD  $(Z_i)$  so that*

- a)  $\text{Sz}(Z) = \text{Sz}(X)$ ,
- b) *if  $X$  is reflexive then  $Z$  is reflexive and  $\text{Sz}(X^*) = \text{Sz}(Z^*)$ , and*

c) if  $X$  has the  $w^*$ -Unconditional Tree Property, then  $(Z_i)$  is unconditional.

The second result uses a construction in [19] and allows to pass from FDDs to bases.

**Theorem B.** *Assume that  $V$  is Banach space with an FDD  $(V_j)$ . Then there exists a Banach space  $W$  with a basis  $(w_j)$ , which contains  $V$  so that*

- a) *if  $(V_i)$  is shrinking, so is  $(w_j)$ , and in that case  $\text{Sz}(W) = \text{Sz}(V)$ ,*
- b) *if  $V$  is reflexive so is  $W$ , and in this case  $\text{Sz}(W^*) = \text{Sz}(V^*)$ , and*
- c) *if  $(V_j)$  is an unconditional FDD then  $(w_j)$  is an unconditional basis.*

It is noteworthy to mention that, independently from the property of the given space  $X$ , the construction of the spaces  $Z$  and  $W$  is the same.  $Z$  and  $W$  inherit automatically the additional properties from  $X$  mentioned in (a), (b) and (c) of the aforementioned Theorems. Since the construction is very concrete one may hope that  $X$  and its superspaces  $Z$  and  $W$  (where  $W$  is built for  $V = Z$ ) share also other properties.

Our paper will be organized as follows. In Section 2 we introduce *Finite Dimensional Markushevich Decompositions* (FMD) of a separable Banach space  $X$ , which are obtained by *blocking* a given Markushevich basis. We finish Section 2 with a blocking Lemma 2.4 which shows that a given FMD can be blocked into a further FMD which has the property that *skipped blocks* are basic sequences. Starting with an appropriately blocked shrinking FMD of the space  $X$  with separable dual  $X^*$  we construct in Section 3 the space  $Z$  with FDD  $(Z_i)$ , which contains  $X$ . Then we prove an FDD version of Zippin's Theorem, namely that  $(Z_i)$  is shrinking (Lemma 3.5), and that  $(Z_i)$  is boundedly complete if  $X$  is reflexive (Lemma 3.7). Moreover we prove that if the biorthogonal sequence  $(F_j)$  of  $(E_j)$  (which is an FMD of  $X^*$ ) is *skipped unconditional*, then  $(Z_j)$  is unconditional. In the second part of Section 3 we construct for a space  $V$  with FDD  $(V_j)$  a space  $W$  containing  $V$  with a basis  $(w_j)$ , which is shrinking if  $(V_j)$  is shrinking, and, moreover, boundedly complete if  $V$  is reflexive, and which is unconditional if  $(V_j)$  is unconditional (Theorem 3.9). We therefore proved Zippin's Theorems 1.1 and 1.2, and, moreover, we reduced the proof of Johnson's and Zheng's results [16, 17] to the problem of showing that the  $w^*$ -UTP implies the existence of FMDs which are skipped unconditional. Section 4 serves as an introduction to Section 5. We introduce certain trees on sets and different ordinal valued indices on them. We also introduce as an example *Schreier* and *Fine Schreier Families*, and observe how these families can serve to measure the indices of trees. At the end of Section 4 we verify some type of *concentration phenomena* for families of functions defined on maximal Schreier sets (Corollary 4.10). In Section 5 we recall the definition of the Szlenk index  $\text{Sz}(K)$  for bounded  $K \subset X^*$  and the Szlenk index of  $X$ , defined by  $\text{Sz}(X) = \text{Sz}(B_{X^*})$ . We recall some, for our purposes relevant, results from the literature. Using the above mentioned Corollary 4.10, we prove the following result on the Szlenk index which is of independent interest:

**Theorem C.** *If  $K \subset B_{X^*}$  is norming  $X$ , then  $\text{Sz}(X) = \min\{\omega^\alpha : \alpha < \omega_1 \text{ and } \omega^\alpha \geq \text{Sz}(K)\}$ .*

With the help of Theorem C we verify the claims on the Szlenk indices in (a) and (b) of Theorems A and B at the end of Section 5. In our last Section 6 we recall *Infinite Asymptotic Games* as introduced in [22, 23] but with respect to FMDs instead of FDDs and show that the main results also hold in this more general framework. We then proof the last part of Theorem A, and show that if  $X$  enjoys the  $w^*$ -UTP a given shrinking FMD can be blocked to be skipped unconditional.

## 2. FINITE DIMENSIONAL MARKUSHEVICH DECOMPOSITIONS

In this section we introduce *Finite Dimensional Markushevich Decompositions* of a separable Banach space  $X$ . These are the multidimensional versions of *Markushevich bases*.

Let  $X$  be a separable Banach space. By a result of Markushevich [20] (see also [12, Theorem 1.22])  $X$  admits a *Markushevich basis*, or *M-basis* which is 1-norming. Recall that a sequence  $(e_i) \subset X$  is called *fundamental for  $X$*  if  $\text{span}(e_i : i \in \mathbb{N})$ , the linear span of  $(e_i)$ , is norm dense in  $X$ , and a fundamental sequence  $(e_i)$  is called *minimal* if  $e_i \notin \overline{\text{span}(e_j : j \in \mathbb{N} \setminus \{i\})}$ , for every  $i \in \mathbb{N}$ . The Hahn Banach Theorem yields that the minimality of a fundamental sequence  $(e_i)$  in  $X$  is equivalent to the existence of a unique sequence  $(f_i) \subset X^*$  which is biorthogonal to  $(e_i)$ . If  $(e_i)$  is fundamental and minimal and  $(f_i)$  is its biorthogonal sequence, we say that  $(f_i)$  is *total*, if for all  $x \in X$ ,  $f_i(x) = 0$ , for all  $i \in \mathbb{N}$ , implies that  $x = 0$ . A fundamental and minimal sequence  $(e_i)$ , whose biorthogonals  $(f_i)$  are total, is called a *Markushevich basis* or *M-basis*. If  $(e_i)$  is a Markushevich basis, the biorthogonal sequence  $(f_j)$  of  $(e_i)$ , is called *c-norming* for some  $c \in (0, 1]$ , if

$$\sup_{f \in \text{span}(f_j : j \in \mathbb{N}), \|f\| \leq 1} f(x) \geq c\|x\|.$$

If  $(e_i)$  is a Markushevich basis and  $(f_j)$  are its biorthogonals, we call a sequence  $(E_k)$  with  $E_k = \text{span}(e_j : n_{k-1} < j \leq n_k)$ , where  $0 = n_0 < n_1 < n_2 < \dots$  are in  $\mathbb{N}$ , a *blocking of  $(e_j)$  into finite dimensional spaces* and note that in that case

- a) the sequence  $(E_k)$  is *fundamental*, i.e.  $\text{span}(E_k : k \in \mathbb{N})$  is dense in  $X$ ,
  - b)  $(E_k)$  is *minimal*, meaning that  $E_k \cap \overline{\text{span}(E_j : j \in \mathbb{N} \setminus \{k\})} = \{0\}$ , for every  $k \in \mathbb{N}$ .
- In that case we call the sequence  $(F_k)$ , with

$$\begin{aligned} F_k &= \text{span}(E_j : j \in \mathbb{N} \setminus \{k\})^\perp \\ &= \{f \in X^* : f|_{\text{span}(E_j : j \in \mathbb{N} \setminus \{k\})} = 0\} = \text{span}(f_j : n_{k-1} < j \leq n_k), \text{ for } k \in \mathbb{N}, \end{aligned}$$

the biorthogonal sequence to  $(E_j)$ .

- c)  $(F_k)$  is *total*, which means that for  $x \in X$ , with  $f(x) = 0$ , for all  $f \in F_k$  and  $k \in \mathbb{N}$ , it follows that  $x = 0$ .
- d) In the case that  $(f_j)$  is *c-norming*, then  $(F_k)$  is also *c-norming*,

$$\|x\| \geq c \sup_{f \in \text{span}(F_j : j \in \mathbb{N}), \|f\| \leq 1} f(x).$$

We call any sequence  $(E_k)$  of finite dimensional subspaces of  $X$  a *Finite Dimensional Markushevich Decomposition of  $X$  (FMD)* if  $(E_k)$  and the sequence  $(F_k)$ , as defined by the first equation in (b), satisfy (a), (b) and (c). As we just pointed out, any blocking of an *M-basis* of  $X$  is an FMD of  $X$ . Conversely, it is also easy to obtain an *M-basis* from an FMD. Indeed, assume that  $(E_k)$  is an FMD and let  $(F_k)$  be its biorthogonal sequence. First note that it follows that  $F_k$  separates points of  $E_k$ , and  $E_k$  separates the points of  $F_k$ , for each  $k \in \mathbb{N}$ , and thus  $\dim(E_k) = \dim(F_k)$ , and we can find a basis  $(e_j^{(k)} : 1 \leq j \leq \dim(E_k))$  of  $E_k$  and a basis  $(f_j^{(k)} : 1 \leq j \leq \dim(E_k))$  of  $F_k$  which is biorthogonal to  $(e_j^{(k)} : 1 \leq j \leq \dim(E_k))$ . It follows therefore that the set  $\{e_j^{(k)} : k \in \mathbb{N}, 1 \leq j \leq \dim(E_k)\}$ , arbitrarily ordered into a sequence, is an *M-basis* of  $X$  and  $\{f_j^{(k)} : k \in \mathbb{N}, 1 \leq j \leq \dim(E_k)\}$  are the biorthogonals.

Assume that  $(E_j)$  is an FMD of  $X$ . From the minimality in (b) it follows that every  $x \in \text{span}(E_j : j \in \mathbb{N})$  can be written uniquely as  $x = \sum_{j=1}^{\infty} x_j$ , with  $x_j \in E_j$ , for  $j \in \mathbb{N}$ , and

$\#\{j \in \mathbb{N} : x_j \neq 0\} < \infty$ , thus we can identify  $\text{span}(E_j : j \in \mathbb{N})$  with

$$c_{00}(E_j) = \{(x_j) : x_j \in E_j, j \in \mathbb{N}, \text{ and } \#\{j \in \mathbb{N} : x_j \neq 0\} < \infty\}.$$

From the minimality condition (b) it also follows for all  $m \in \mathbb{N}$ , that  $X$  is the complemented sum of  $E_m$  and the space

$$\overline{\text{span}(E_n : n \in \mathbb{N} \setminus \{m\})} = {}^\perp F_m = \{x \in X : x^*(x) = 0 \text{ for all } x^* \in F_m\}.$$

Thus, for every  $m \in \mathbb{N}$  the projection  $P_m^E : X \rightarrow E_m$  is bounded, where  $P_m^E(x) = x_m$ , for  $x \in X$ , if  $x = y_m + x_m$  is the unique decomposition of  $x$  into  $y_m \in \overline{\text{span}(E_n : n \in \mathbb{N} \setminus \{m\})}$  and  $x_m \in E_m$ . For a finite set  $A \subset \mathbb{N}$  we define  $P_A^E = \sum_{m \in A} P_m^E$  and for a cofinite  $A \subset \mathbb{N}$  we put  $P_A^E = Id - \sum_{m \in \mathbb{N} \setminus A} P_m^E$ . For  $x \in X$  we call the *support of  $x$  with respect to  $(E_n)$*  the set

$$\text{supp}_E(x) = \{j \in \mathbb{N} : P_j^E(x) \neq 0\}.$$

For  $x^* \in X^*$  we define the *support of  $x^*$  with respect to  $(E_n)$*  by

$$\text{supp}_E(x^*) = \{j \in \mathbb{N} : x^*|_{E_j} \neq 0\}.$$

The *range of  $x \in X$  or  $x^* \in X^*$*  is the smallest interval in  $\mathbb{N}$  containing the support of  $x$ , or  $x^*$ , and is denoted by  $\text{rg}_E(x)$ , or  $\text{rg}_E(x^*)$ . A *block sequence with respect to  $(E_n)$  in  $X$  or in  $X^*$*  is a finite or infinite sequence  $(x_n)$  in  $X$ , or a sequence  $(x_n^*)$  in  $X^*$  for which  $\max \text{rg}_E(x_n) < \min \text{rg}_E(x_{n+1})$  or  $\max \text{rg}_E(x_n^*) < \min \text{rg}_E(x_{n+1}^*)$ , respectively, for all  $n \in \mathbb{N}$  for which  $x_{n+1}$ , or  $x_{n+1}^*$  are defined. In the case that  $\max \text{rg}_E(x_n) < \min \text{rg}_E(x_{n+1}) - 1$  or  $\max \text{rg}_E(x_n^*) < \min \text{rg}_E(x_{n+1}^*) - 1$ , respectively, we call the sequence a *skipped block sequence with respect to  $(E_n)$  in  $X$  or in  $X^*$* . Note that in finite blocks the last element does not need to have a finite range.

It is easy to see that the sequence  $(F_j)$  is an FMD of  $Y = \overline{\text{span}(F_j : j \in \mathbb{N})}$  whose biorthogonal sequence is  $(E_j)$ . Here we identify  $X$  in the canonical way with a subspace of  $Y^*$ . Using our notation it follows then for  $y \in Y$ , that  $\text{supp}_F(y) = \text{supp}_E(y)$ , and  $\text{rg}_F(y) = \text{rg}_E(y)$ . But since we want to apply the support and range also to elements of  $X^*$  which are not in  $Y$ , we prefer to write  $\text{rg}_E(y)$ , and  $\text{supp}_E(y)$ .

Similar to the case of  $M$ -bases we can of course also define blockings of an FMD  $(E_j)$  as follows. If  $(E_j)$  is an FMD and  $(F_j)$  is its biorthogonal sequence, then  $(G_k)$  is a *blocking of  $(E_j)$*  if  $G_k = \text{span}(E_j : n_{k-1} < j \leq n_k)$ , for all  $k \in \mathbb{N}$ , and some natural numbers  $0 = n_0 < n_1 < n_2 \dots$ .  $(G_k)$  is then also an FMD of  $X$  and its biorthogonal sequence is  $(H_k)$  with  $H_k = \text{span}(F_j : n_{k-1} < j \leq n_k)$ , and  $(H_k)$  is  $c$ -norming if  $(F_j)$  was  $c$ -norming.

An FMD  $(E_j)$  is called a *Finite Dimensional Decomposition of  $X$*  or FDD, if for every  $x \in X$  there is a unique sequence  $(x_j)$ ,  $x_j \in E_j$ , for  $j \in \mathbb{N}$ , so that  $x = \sum_{j=1}^{\infty} x_j$ . As in the case of Schauder bases it follows from the Uniform Boundedness Principle that an FMD of  $X$  is an FDD of  $X$  if and only if the sequence  $(P_{[1,k]}^E : k \in \mathbb{N})$  is uniformly bounded. As in the case of Schauder bases we call for an FDD  $(E_n)$  the number  $b = \sup_{m \leq n} \|P_{[m,n]}^E\|$  the *projection constant of  $(E_j)$* , and we call  $(E_j)$  *bimonotone* if  $b = 1$ . We call an FDD  $(E_i)$  *shrinking* if the biorthogonal sequence  $(F_n)$  spans a dense subspace of  $X^*$ , and we call  $(E_i)$  *boundedly complete* if for every block sequence  $(x_n)$ , for which  $\sup_{n \in \mathbb{N}} \|\sum_{j=1}^n x_j\| < \infty$ , the series  $\sum_{j=1}^{\infty} x_j$  converges. An FDD  $(E_j)$  is called *unconditional* if

$$c_u = \sup \left\{ \left\| \sum_{j=1}^{\infty} \sigma_j x_j \right\| : (\sigma_j) \in \{\pm 1\}^{\omega}, \left\| \sum_{j=1}^{\infty} x_j \right\| \leq 1, x_j \in E_j, j \in \mathbb{N} \right\} < \infty.$$

This is equivalent with

$$c_s = \sup \{ \|P_A^E\| : A \subset \mathbb{N}, \text{ finite} \} < \infty,$$

and in this case  $c_s \leq c_u \leq 2c_s$ . An FDD  $(E_n)$  is called  $c$ -unconditional if  $c_u \leq c$  and  $c$ -suppression unconditional if  $c_s \leq c$ .

We avoid to denote the biorthogonal sequence  $(F_n)$  of an FMD  $(E_n)$  by  $(E_n^*)$  because we reserve the notion  $E^*$  to the dual space of a space  $E$ . Of course the map  $T_n : F_n \rightarrow E_n^*$ ,  $x^* \mapsto x^*|_{E_n}$  is a linear bijection, and  $\|T_n\| \leq 1$ , for  $n \in \mathbb{N}$ , but, unless  $(F_n)$  is an FDD the inverses of the  $T_n$  may not be uniformly bounded. Nevertheless, for Markushevich bases  $(e_n)$  we will denote, as usual, the biorthogonals by  $(e_n^*)$ . Also in case that  $(E_n)$  is an FDD we will denote the biorthogonal sequence by  $(E_n^*)$ .

As in the case of bases or FDDs, we call an FMD  $(E_j)$  *shrinking in  $X$*  if the span of the biorthogonal sequence  $(F_j)$  is dense in  $X^*$ . Note that in this case  $(F_j)$  is an FMD of  $X^*$  whose biorthogonal sequence is  $(E_j)$ . Recall that if  $X^*$  is separable then  $X$  admits a shrinking  $M$ -basis [12, Lemma 1.21].

The proof of the following observation is obtained like in the case of  $M$ -bases or in the case of FDDs.

**Proposition 2.1.** *Assume that  $(E_n)$  is an FMD of  $X$ . The following are equivalent:*

- (1)  $(E_n)$  is shrinking,
- (2) for all  $x^* \in X^*$  it follows that  $\lim_{n \rightarrow \infty} \|x^*|_{\text{span}(E_j : j > n)}\| = 0$ ,
- (3) every bounded sequence  $(y_n)$ , with  $y_n \in \text{span}(E_j : j \in \mathbb{N}, j \geq n)$ , is weakly null.

*Remark 2.2.* From the equivalence (1)  $\iff$  (3) in Proposition 2.1 we deduce that in a reflexive space  $X$  every FMD  $(E_n)$  of  $X$  is shrinking, and thus the biorthogonal sequence  $(F_j)$  is a shrinking FMD of  $X^*$ . Indeed, if  $y_n \in B_X \cap \text{span}(E_j : j \in \mathbb{N}, j \geq n)$ , for  $n \in \mathbb{N}$ , then we can assume that  $y_n$  is weakly converging to some  $y \in X$ , but  $y^*(y)$  must vanish for all  $y^* \in \text{span}(F_j : j \in \mathbb{N})$  and it follows therefore that  $y = 0$ .

*Proof of Proposition 2.1.* Let  $(F_n)$  be the biorthogonal sequence of  $(E_n)$ .

“(1)  $\Rightarrow$  (2)” If  $(E_n)$  is shrinking and  $x^* \in X^*$  we can find for an arbitrary  $\varepsilon > 0$  an element  $y^* = \sum f_j \in \text{span}(F_j : j \in \mathbb{N})$ ,  $f_j \in F_j$ , for  $j \in \mathbb{N}$ , so that  $\|x^* - y^*\| < \varepsilon$ . Thus

$$\limsup_{n \rightarrow \infty} \|x^*|_{\text{span}(E_j : j > n)}\| \leq \varepsilon + \lim_{n \rightarrow \infty} \|y^*|_{\text{span}(E_j : j > n)}\| = \varepsilon,$$

which proves our claim since  $\varepsilon > 0$  is arbitrary.

“(2)  $\Rightarrow$  (1)” Assume (2) is satisfied and let  $x^* \in X^*$  and  $\varepsilon > 0$ . For large enough  $m$  it follows that  $\|x^*|_{\text{span}(E_j : j > m)}\| < \varepsilon$ . Now let  $y^* \in X^*$  be a Hahn-Banach extension of  $x^*|_{\text{span}(E_j : j > m)}$ . Then  $x^* - y^* \in \text{span}(F_j : j \geq m)$  (since  $(x^* - y^*)|_{\text{span}(E_j : j < m)} \equiv 0$ ) and  $\|x^* - (x^* - y^*)\| = \|y^*\| < \varepsilon$ .

“(3)  $\Rightarrow$  (2)” Assume  $x_n \in B_X \cap \text{span}(E_j : j > n)$  and  $(x_n)$  is not weakly null. After passing to a subsequence we can assume that there is an  $x^* \in B_{X^*}$  with  $|x^*(x_n)| \geq \varepsilon > 0$ . Then

$$\|x^*|_{\text{span}(E_j : j > n)}\| \geq |x^*(x_n)| \geq \varepsilon,$$

thus (2) is not satisfied.

“(2)  $\Rightarrow$  (3)” Assume that  $\|x^*|_{\text{span}(E_j : j > n)}\| \geq \varepsilon > 0$  for all  $n$ . Then choose  $x_n \in B_X \cap \text{span}(E_j : j > n)$  with  $|x^*(x_n)| \geq \varepsilon/2$ . Thus (3) is not satisfied.  $\square$

We finish this introductory section with the following easy, and in similar versions well known observation, which will be crucial for future arguments.

**Lemma 2.3.** *Let  $X$  be a separable Banach space. Assume that  $(E'_j)$  is a 1-norming FMD of  $X$  and  $(F'_j)$  is its biorthogonal sequence. Then  $(E'_j)$  can be blocked to an FMD  $(E_n)$  satisfying with its biorthogonal sequence  $(F_n)$  the following conditions for every  $m \leq n$  in  $\mathbb{N}$ .*

- (1) *For all  $e^* \in (E_m + E_{m+1} + \dots + E_n)^*$  there exists  $x^* \in F_{m-1} + F_m + \dots + F_{n+1}$  so that*

$$x^*|_{E_m + E_{m+1} + \dots + E_n} = e^* \text{ and } \|x^*\| \leq 2.5\|e^*\|,$$

- (2) *for all  $f^* \in (F_m + F_{m+1} + \dots + F_n)^*$  there exists  $z \in E_{m-1} + E_m + \dots + E_{n+1}$  so that*

$$z|_{F_m + F_{m+1} + \dots + F_n} = f^* \text{ and } \|z\| \leq 2.5\|f^*\|,$$

- (3) *for all  $x^* \in F_m + F_{m+1} + \dots + F_n$*

$$\|x^*\| \leq 2.5\|x^*|_{E_{m-1} + E_m + \dots + E_n + E_{n+1}}\| = 2.5 \sup_{x \in E_{m-1} + E_m + \dots + E_{n+1}, \|x\| \leq 1} |x^*(x)|,$$

- (4) *for all  $x \in E_m + E_{m+1} + \dots + E_n$*

$$\|x\| \leq 2.5\|x|_{F_{m-1} + F_m + \dots + F_n + F_{n+1}}\| = 2.5 \sup_{x^* \in F_{m-1} + F_m + \dots + F_n + F_{n+1}, \|x^*\| \leq 1} |x^*(x)|.$$

Here we let  $E_0$  and  $F_0$  be the null spaces in  $X$  and  $X^*$ , respectively.

The proof will follow using repeatedly the following Lemma.

**Lemma 2.4.** *Let  $X$  be a Banach space and let  $Y'$  be a (not necessarily closed) subspace of  $X^*$  for which  $B_{Y'}$  is  $w^*$ -dense in  $B_{X^*}$ . Assume that  $E \subset X$  is finite dimensional, and  $\varepsilon > 0$ . Then there is a finite dimensional subspace  $F \subset Y'$ , so that every  $e^* \in E^*$  can be extended to an element  $x^* \in F$ , with  $\|x^*\| \leq (1 + \varepsilon)\|e^*\|$ .*

*Proof.* Let  $\delta \in (0, \frac{1}{2})$ , and choose a  $\delta$ -net  $(e_j^*)_{j=1}^N$  in  $S_{E^*}$ , and let  $x_j^* \in S_{X^*}$  be a Hahn-Banach extension of  $e_j^*$ , for  $j = 1, 2, \dots, N$ . By the assumption that  $B_{Y'}$  is  $w^*$ -dense in  $B_{X^*}$  we can choose  $(y_j^*)_{j=1}^N \subset B_{Y'}$ , so that  $\|y_j^*|_E - e_j^*\| < \delta$  for all  $j = 1, 2, \dots, N$ . Let  $F = \text{span}(y_j^* : j = 1, 2, \dots, N)$  and consider the restriction map  $T : F \rightarrow E^*$ ,  $x^* \mapsto x^*|_E$ . By our construction  $T(B_F)$  is  $2\delta$ -dense in  $B_{E^*}$ .

Now let  $e^* \in B_{E^*}$ . We can successively choose  $x_1^*, x_2^*, x_3^*, \dots$  so that  $\|x_n^*\| \leq (2\delta)^{n-1}$  and  $\|e^* - (T(x_1^*) + T(x_2^*) + \dots + T(x_{n-1}^*)) - T(x_n^*)\| \leq 2\delta\|(e^* - (T(x_1^*) + T(x_2^*) + \dots + T(x_{n-1}^*)))\| \leq (2\delta)^n$ ,

and thus, letting  $x^* = \sum_{n=1}^{\infty} x_n^* \in F$ , we deduce that  $e^* = T(x^*)$  and  $\|x^*\| \leq \frac{1}{1-2\delta}$ . Choosing  $\delta > 0$  sufficiently small, we obtain our claim.  $\square$

*Proof of Lemma 2.3.* Define  $X' = \text{span}(E'_j : j \in \mathbb{N})$  and  $Y' = \text{span}(F'_j : j \in \mathbb{N})$ . Since  $(E'_j)$  is a 1-norming FMD,  $B_{Y'}$  is  $w^*$ -dense in  $B_{X^*}$ , and moreover the map  $T : X \rightarrow Y^*$ , defined by  $T(x)(y) = y(x)$ , for  $x \in X$  and  $y \in Y$ , is an isometric embedding. It follows that  $B_X$  and, therefore also  $B_{X'}$ , is  $w^*$ -dense in  $B_{Y^*}$ . Let  $\rho > 1$  with  $\rho + \rho^2 < 2.5$ . Inductively, we choose  $0 = n_0 < n_1 < n_2 < \dots$  in  $\mathbb{N}$ , so that for all  $k \in \mathbb{N}_0$  the following two conditions hold.

- (5) For all  $e^* \in (E'_1 + E'_2 + \dots + E'_{n_k})^*$  there is an  $x^* \in F'_1 + F'_2 + \dots + F'_{n_{k+1}}$  so that

$$x^*|_{E'_1 + E'_2 + \dots + E'_{n_k}} = e^* \text{ and } \|x^*\| \leq \rho\|e^*\|,$$

- (6) for all  $f^* \in (F'_1 + F'_2 + \dots + F'_{n_k})^*$  there is an  $x \in E'_1 + E'_2 + \dots + E'_{n_{k+1}}$  so that

$$x|_{F'_1 + F'_2 + \dots + F'_{n_k}} = f^* \text{ and } \|x\| \leq \rho\|f^*\|$$

(with  $E'_1 + E'_2 + \dots + E'_0 = \{0\}$  and  $F'_1 + F'_2 + \dots + F'_0 = \{0\}$ ).

For  $k = 0$  we choose  $n_1 = 1$  and note that (5) and (6) are trivially satisfied.

Assume that we have chosen  $n_k$  for some  $k \geq 1$ . We first apply Lemma 2.4 to  $E = E'_1 + E'_2 + \dots + E'_{n_k}$  and  $Y'$  to obtain a finite dimensional subspace  $F' \subset Y'$  so that every  $e^* \in (E'_1 + E'_2 + \dots + E'_{n_k})^*$  can be extended to an element  $x^* \in F'$ , with  $\|x^*\| \leq \rho \|e^*\|$ . Then we apply Lemma 2.4 to the Banach space  $Y = \overline{Y'}$ , instead of  $X$ , to  $X'$  (recall that  $B_{X'}$  is  $w^*$  dense in  $B_{Y^*}$ ), and to  $F = F'_1 + F'_2 + \dots + F'_{n_k}$  to obtain a finite dimensional subspace  $E'$  of  $X'$  so that every  $f^* \in (F'_1 + F'_2 + \dots + F'_{n_k})^*$  can be extended to an element  $x \in E'$  with  $\|x\| \leq \rho \|f^*\|$ .

Because  $E'$  and  $F'$  are finite dimensional subspaces of  $X'$  and  $Y'$ , respectively, there is some  $n_{k+1} > n_k$  in  $\mathbb{N}$  so that  $E' \subset E'_1 + E'_2 + \dots + E'_{n_{k+1}}$  and  $F' \subset F'_1 + F'_2 + \dots + F'_{n_{k+1}}$ . This finishes the recursive definition of  $n_k$ ,  $k \in \mathbb{N}$ .

We define  $E_k = E'_{n_{k-1}+1} + E'_{n_{k-1}+2} + \dots + E'_{n_k}$  and  $F_k = F'_{n_{k-1}+1} + F'_{n_{k-1}+2} + \dots + F'_{n_k}$ .

In order to verify (1) let  $m \leq n$  and  $e^* \in (E_m + E_{m+1} + \dots + E_n)^*$ . We first apply (5) to obtain  $z^* \in F_1 + F_2 + \dots + F_{n+1}$  which is an extension of  $e^*$  with  $\|z^*\| \leq \rho \|e^*\|$ . If  $m \leq 2$  we can choose  $x^* = z^*$ . Otherwise we apply again (5) to  $z^*|_{E_1 + E_2 + \dots + E_{m-2}}$  and extend it to an element  $y^*$  in  $F_1 + F_2 + \dots + F_{m-1}$  with  $\|y^*\| \leq \rho \|z^*|_{E_1 + E_2 + \dots + E_{m-2}}\| \leq \rho^2 \|e^*\|$  and finally put  $x^* = z^* - y^*$ . Since  $x^*$  vanishes on all  $E_j$  with  $j \in \mathbb{N} \setminus [m-1, n+1]$ , it follows that  $x^* \in F_{m-1} + F_m + \dots + F_n + F_{n+1}$ . It is also clear that  $x^*$  extends  $e^*$  and since  $\|x^*\| \leq \|z^*\| + \|y^*\| \leq \rho \|e^*\| + \rho^2 \|e^*\|$ , we deduce (1). The verification of (2) can be accomplished similarly to the proof of (1).

To show (3) let  $x^* \in F_m + F_{m+1} + \dots + F_n$  and let  $\eta > 0$ . We can choose  $x \in S_X$  so that  $|x^*(x)| \geq \|x^*\| - \eta$ . We view  $x$  as an element of  $Y^*$  and put

$$f^* = x|_{F_m + F_{m+1} + \dots + F_n} \in (F_m + F_{m+1} + \dots + F_n)^*.$$

Using (2) we can extend  $f^*$  to an element  $z \in E_{m-1} + E_m + \dots + E_n + E_{n+1}$ , with  $\|z\| \leq 2.5$  and thus

$$\|x^*\| - \eta \leq |x^*(x)| = |x^*(z)| \leq (2.5) \sup_{y \in E_{m-1} + E_m + \dots + E_n + E_{n+1}, \|y\| \leq 1} |x^*(y)|,$$

which implies our claim since  $\eta > 0$  was arbitrary.

(4) can be shown the same way using (1). □

From (3) we easily observe the following

**Corollary 2.5.** *Assume the sequence  $(E_n)$  is an FMD of  $X$  with biorthogonal sequence  $(F_j)$  satisfying the conclusions of Lemma 2.3. Then every skipped block in  $X$  and block sequence  $(y_j)$  in  $Y = \text{span}(F_j : j \in \mathbb{N})$  is basic with a projection constant not larger than 2.5.*

### 3. CONSTRUCTION OF $Z$ AND $W$ AND PROOF OF ZIPPIN'S THEOREMS

Throughout this section  $X$  is a Banach space whose dual  $X^*$  is separable. We also assume that we have chosen a shrinking FMD  $(E_n)$  of  $X$  which, together with its biorthogonal sequence  $(F_n)$ , satisfies the conclusions of Lemma 2.3. The following observation was made in [15] in the FDD case. The proof in the FMD case is the same.

**Lemma 3.1.** *Let  $(\varepsilon_k) \subset (0, 1]$  be given. Then there is an increasing sequence  $(n_k) \subset \mathbb{N}$ , so that for each  $x^* \in B_{X^*}$  and each  $k \in \mathbb{N}$  there is a  $j_k \in [n_k, n_{k+1}]$ , so that  $\|x^*|_{E_{j_k}}\| \leq \varepsilon_k$ .*

*Proof.* Our conclusion follows from iterating the following claim.

**Claim:** for any  $\delta > 0$  and any  $m \in \mathbb{N}$  there is an  $n > m$  in  $\mathbb{N}$  so that for each  $x^* \in B_{X^*}$  there is a  $j \in [m, n]$  with  $\|x^*|_{E_j}\| \leq \delta$ .



Assume that our claim is not true, and for each  $n \geq m$  we could choose  $x_n^* \in B_{X^*}$  so that  $\|x_n^*|_{E_j}\| \geq \delta$  for all  $j \in [m, n]$ . After passing to a subsequence we can assume that  $(x_n^*)$   $w^*$ -converges to some  $x^* \in B_{X^*}$ . But then it follows that  $\|x^*|_{E_j}\| \geq \delta$ , for all  $j \geq m$  which contradicts property (2) in Proposition 2.1.  $\square$

We choose a decreasing sequence  $(\varepsilon_k) \subset (0, 1)$  with  $\sum_{k \in \mathbb{N}} \varepsilon_k < \frac{1}{50}$  and let  $(n_k) \subset \mathbb{N}$  satisfy the conclusion of Lemma 3.1. Note that this choice implies that  $n_{k+1} > n_k + 2$ , for  $k \in \mathbb{N}$ . Then we define

$$(7) \quad D^* = \{x^* \in X^* : \forall k \in \mathbb{N} \exists j \in [n_k, n_{k+1}] \quad x^*|_{E_j} \equiv 0\}, \text{ and } B^* = D^* \cap B_{X^*}.$$

**Lemma 3.2.**  $B^*$  is  $\frac{1}{10}$ -dense in  $B_{X^*}$ .

*Proof.* Let  $x^* \in B_{X^*}$ , and choose, according to Lemma 3.1,  $j_k \in [n_k, n_{k+1}]$ , for each  $k \in \mathbb{N}$ , so that  $\|x^*|_{E_{j_k}}\| \leq \varepsilon_k$ . In the case that for some  $k \in \mathbb{N}$  we have  $j_{k+1} = j_k + 1$  we change  $j_k$  and  $j_{k+1}$  in the following way: First note that  $j_{k+1} = j_k + 1$  only happens if  $j_k = n_{k+1} - 1$  and  $j_{k+1} = n_{k+1}$  or  $j_k = n_{k+1}$  and  $j_{k+1} = n_{k+1} + 1$ . In that case we redefine  $j_k = j_{k+1} = n_{k+1}$ . Then we define  $K = \{k \in \mathbb{N} : j_k \neq j_{k-1}\}$ . This, and the above observed fact, that  $n_{k+1} > n_k + 2$ , for  $k \in \mathbb{N}$ , implies that  $(j_k : k \in K)$  is a skipped sequence in  $K$ , and we still have  $j_k \in [n_k, n_{k+1}]$  and  $\|x^*|_{E_{j_k}}\| \leq \varepsilon_k$ , for each  $k \in K$ .

Applying for each  $k \in K$ , part (1) of Lemma 2.3 to  $e_k^* = x^*|_{E_{j_k}}$ , we obtain an extension of  $e_k^*$  to  $f_k \in F_{j_k-1} + F_{j_k} + F_{j_k+1}$  with  $\|f_k\| \leq 2.5\varepsilon_k$ . Then we define

$$y^* = x^* - \sum_{k \in K} f_k \text{ and } z^* = \frac{20}{21}y^*.$$

Since  $\|y^*\| \leq \|x^*\| + \frac{1}{20} \leq \frac{21}{20}$  it follows that  $\|z^*\| \leq 1$  and thus  $z^* \in B^*$  and

$$\|x^* - z^*\| = \|x^* - y^*\| + \left\|y^* - \frac{20}{21}y^*\right\| \leq \|x^* - y^*\| + \frac{1}{20} < \frac{1}{10},$$

which finishes the proof of our claim.  $\square$

From now on we assume, possibly after renorming  $X$ , that

$$(8) \quad \|x\| = \sup_{x^* \in B^*} |x^*(x)| \text{ for all } x \in X,$$

in other words we assume that  $B^* \subset B_{X^*}$ , as defined in (7), is 1-norming the space  $X$ .

For  $x^* \in D^*$  we let  $\bar{j} = (j_k) \in \prod_{k=1}^{\infty} [n_k, n_{k+1}]$  so that  $x^*|_{E_{j_k}} \equiv 0$  and put  $x_k^* = P_{(j_{k-1}, j_k)}^F(x^*)$  for  $k \in \mathbb{N}$  (with  $j_0 = 0$ ). Since  $(E_j)$  is shrinking it follows from Proposition 2.1 and Lemma 2.3 for  $m \in \mathbb{N}$  that

$$\left\|x^* - \sum_{k=1}^m x_k^*\right\| \leq 3 \sup_{x \in \text{span}(E_j : j \geq j_m), \|x\| \leq 1} \left| \left(x^* - \sum_{k=1}^m x_k^*\right)(x) \right| = 3 \|x^*|_{\text{span}(E_j : j \geq j_m)}\| \rightarrow_{m \rightarrow \infty} 0.$$

Thus

$$(9) \quad x^* = \sum_{k=1}^{\infty} x_k^* \text{ and this series converges in norm, for all } x^* \in D^*.$$

Lemma 2.3 also yields for  $m \leq n$  that

$$(10) \quad \left\| \sum_{k=m}^n x_k^* \right\| \leq 3 \sup_{x \in E_{j_{m-1}} + \dots + E_{j_n}, \|x\| \leq 1} \sum_{k=m}^n x_k^*(x)$$

$$= 3 \sup_{x \in E_{j_{m-1}} + \dots + E_{j_n}, \|x\| \leq 1} \sum_{k=1}^{\infty} x_k^*(x) \leq 3\|x^*\|.$$

We define

$$(11) \quad \mathbb{D}^* = \left\{ (x_k^*) \subset X^* : \begin{array}{l} \exists (j_k) \in \prod_{k=1}^{\infty} [n_k, n_{k+1}] \text{ so that} \\ \text{rg}_E(x_k^*) \subset (j_{k-1}, j_k), \text{ for } k \in \mathbb{N}, \left\| \sum_{k=1}^{\infty} x_k^* \right\| < \infty \end{array} \right\}$$

(In the definition of  $\mathbb{D}^*$  it is possible that  $j_{k-1} = j_k = n_k$  or  $j_k = j_{k-1} + 1$ , and that in either case  $x_k^* \equiv 0$ ) and

$$(12) \quad \mathbb{B}^* = \mathbb{D}^* \cap \left\{ (x_k^*) \subset X^* : \left\| \sum_{k=1}^{\infty} x_k^* \right\| \leq 1 \right\}.$$

We can rewrite the sets  $D^*$  and  $B^*$  as

$$(13) \quad D^* = \left\{ \sum_{k=1}^{\infty} x_k^* : (x_k^*) \in \mathbb{D}^* \right\} \text{ and } B^* = D^* \cap B_{X^*} = \left\{ \sum_{k=1}^{\infty} x_k^* : (x_k^*) \in \mathbb{B}^* \right\}.$$

We now construct the space  $Z$  with shrinking FDD  $(Z_j)$  which contains  $X$ . The important point in the construction of the space will be that  $\mathbb{B}^*$  will become the 1-norming set of  $Z$ , and that the similarities between  $X$  and  $Z$  stem from the similarities of the sets  $D^*$  and  $\mathbb{D}^*$ , and  $B^*$  and  $\mathbb{B}^*$ , respectively. We put for  $k \in \mathbb{N}$

$$(14) \quad Z_k = \text{span}(E_j : n_{k-1} < j < n_{k+1}), \text{ for } k \in \mathbb{N} \text{ (as before, } n_0 = 0),$$

and note that for  $(x_k^*) \in \mathbb{B}^*$  and each  $k \in \mathbb{N}$  it follows that  $\text{rg}_E(x_k^*) \subset (n_{k-1}, n_{k+1})$ ,  $x_k^*$  can therefore be seen as functional acting on  $Z_k$ . For  $z = (z_k) \in c_{00}(Z_k)$  we define

$$(15) \quad \|z\| = \sup \left\{ \sum_{k=1}^{\infty} x_k^*(z_k) : (x_k^*) \in \mathbb{B}^* \right\}.$$

We define  $Z$  to be the completion of  $c_{00}(Z_k)$  with respect to  $\|\cdot\|$ . We will from now on consider the elements of  $\mathbb{D}^*$  to be elements of  $Z^*$ . If  $x^* \in X^*$  with  $\text{rg}_E(x^*) \subset (n_{k-1}, n_{k+1})$ , for some  $k \in \mathbb{N}$ , we can identify  $x^*$  with the sequence  $(x_m^*) \in \mathbb{D}^*$ , with  $x_m^* = x^*$ , if  $m = k$ , and  $x_m^* = 0$ , otherwise. Thus we can consider  $x^*$  to be an element of  $Z^*$ .

The following Proposition gathers some properties of the space  $Z$ , and shows how  $Z$  inherits the properties of  $X$ .

**Proposition 3.3.** (*Properties of the space  $Z$* )

(1) *The map*

$$I : c_{00}(E_j) \rightarrow Z, \quad x \mapsto (P_{(n_{k-1}, n_{k+1})}^E(x) : k \in \mathbb{N})$$

*extends to an isometric embedding from  $X$  into  $Z$ .*

(2)  *$(Z_j)$  is an FDD of  $Z$  whose projection constant is not larger than 3.*

(3) *For a sequence  $\bar{j} = (j_k) \in \prod_{k=1}^{\infty} [n_k, n_{k+1}]$ , define*

$$U_{\bar{j}} = \{x^* \in X^* : \forall k \in \mathbb{N}, x^*|_{E_{j_k}} \equiv 0\}.$$

*Then  $U_{\bar{j}}$  is a  $w^*$ -closed subspace of  $X^*$  and the map*

$$\Phi_{\bar{j}} : U_{\bar{j}} \rightarrow Z^*, \quad x^* \mapsto (P_{(j_{k-1}, j_k)}^F(x^*) : k \in \mathbb{N}),$$

*is an isometric embedding which is continuous with respect to the  $w^*$ -topology of  $X^*$  restricted to  $U_{\bar{j}}$  and the  $w^*$ -topology of  $Z^*$ .*

- (4)  $\mathbb{B}^*$  is a  $w^*$ -compact subset of  $B_{Z^*}$  which is 1-norming  $Z$  and the restriction of  $I^* : Z^* \rightarrow X^*$  to the set  $\mathbb{D}^*$  is a norm preserving map from  $\mathbb{D}^*$  onto  $D^*$ .

Moreover, if  $(z_i^*)$  is a skipped block with respect to the FDD  $(Z_k^*)$  whose elements are in  $\mathbb{D}^*$ , then  $(I^*(z_i^*))$  is a skipped block in  $D^*$  with respect to  $(F_j)$  which is isometrically equivalent to  $(z_i^*)$ .

- (5) Let  $Y$  be a Banach space and let  $T_k : Y \rightarrow Z_k$ , be linear, for  $k \in \mathbb{N}$ , and assume that  $T_k \equiv 0$  for all but finitely many  $k$ . Define

$$T : Y \rightarrow Z, \quad y \mapsto (T_k(y) : k \in \mathbb{N}).$$

For every  $\bar{x}^* = (x_k^*)_{k=1}^\infty \in \mathbb{B}^*$  define

$$T_{\bar{x}^*}^* = T^*|_{\overline{\text{span}(x_k^* : k \in \mathbb{N})}} : \overline{\text{span}(x_k^* : k \in \mathbb{N})} \rightarrow Y^*, \quad \sum a_k x_k^* \mapsto \sum a_k x_k^* \circ T_k.$$

Then

$$(16) \quad \|T\|_{L(Y, Z)} = \sup_{\bar{x} \in \mathbb{B}^*} \|T_{\bar{x}^*}^*\|.$$

We expressed therefore the norm of an operator  $T : X \rightarrow Z$  by the norm of its adjoint restricted to spaces of the form  $\overline{\text{span}(x_j^* : j \in \mathbb{N})}$  with  $(x_j^*) \in \mathbb{B}^*$ .

*Proof.* To verify (1) let  $x \in c_{00}(E_j)$  and note that

$$\|I(x)\| = \sup_{(x_k^*) \in \mathbb{B}^*} \left| \sum_{k=1}^\infty x_k^*(P_{(n_{k-1}, n_{k+1})}^E(x)) \right| = \sup_{(x_k^*) \in \mathbb{B}^*} \left| \sum_{k=1}^\infty x_k^*(x) \right| = \sup_{x^* \in B^*} |x^*(x)| = \|x\|.$$

- (2) Let  $m \leq n$  and  $z = (z_k) \in c_{00}(Z_k)$ . Then it follows from (10)

$$\|P_{[m, n]}^Z(z)\| = \sup_{(x_k^*) \in \mathbb{B}^*} \left\| \sum_{k=m}^n x_k^* z_k \right\| \leq \sup_{(x_k^*) \in \mathbb{B}^*} \left\| \sum_{k=m}^n x_k^* \right\| \cdot \|z\| \leq 3\|z\|.$$

- (3) For  $\bar{j} = (j_k) \in \prod_{k=1}^\infty [n_k, n_{k+1}]$  the space  $U_{\bar{j}} = \{x^* \in X^* : \forall k \in \mathbb{N}, x^*|_{E_{j_k}} \equiv 0\}$  is clearly a  $w^*$ -closed subspace of  $X^*$ . If  $x^* \in U_{\bar{j}}$ , with  $\|x^*\| = 1$  put  $x_k^* = P_{(j_{k-1}, j_k)}^F(x^*)$ ,  $k \in \mathbb{N}$ , and  $z^* = (x_k^*)$ . On the one hand it follows from the definition of the norm on  $Z$  that  $z^* \in \mathbb{B}^*$  and, thus,  $\|z^*\| \leq 1$ . On the other hand it follows for all  $x \in X$  that  $z^*(I(x)) = \sum_{k=1}^\infty x_k^*(x) = x^*(x)$  and thus, since  $I$  is an isometric embedding,  $\|z^*\| \geq \|x^*\| = 1$ . Thus  $\Phi_{\bar{j}}$  is an isometric embedding from  $U_{\bar{j}}$  into  $Z^*$ .

In order to show that  $\Phi_{\bar{j}}$  is  $w^*$ -continuous, it is enough to show that  $\Phi_{\bar{j}}$  restricted to the unit ball is  $w^*$  continuous. Then the  $w^*$ -continuity on all of  $U_{\bar{j}}$  follows from the already observed fact that  $U_{\bar{j}}$  is  $w^*$ -closed, and the Theorem of Krein-Smulian, which says that a subspace a dual space is  $w^*$  closed if its intersection with the unit ball is  $w^*$ -closed (c.f. [6, V.5 Corollary 8])

In order to show that  $\Phi_{\bar{j}}$  is  $w^*$ -continuous on  $B_{U_{\bar{j}}}$ , let  $(x^*(n))$  be a sequence in  $B_{U_{\bar{j}}} \subset B_{X^*}$  which  $w^*$ -converges to  $x^*$ , and let  $x_k^* = P_{(j_{k-1}, j_k)}^F(x^*)$  and  $x_k^*(n) = P_{(j_{k-1}, j_k)}^F(x^*(n))$ , for  $k, n \in \mathbb{N}$ . It follows that for each  $k \in \mathbb{N}$  the sequence  $x_k^*(n)$  converges to  $x_k^*$  in  $Z_k^*$  and thus  $(x_k^*(n))$  converge in  $w^*$  to  $x_k^*$  as functionals on  $Z^*$ , which act on the  $k$ -th coordinate. But this implies that the sequence  $\Psi_{\bar{j}}(x^*(n))$  converges point wise on a dense set of  $Z$  to  $\Phi_{\bar{j}}(x^*)$ . Since  $(x^*(n) : n \in \mathbb{N})$  is bounded we deduce that  $(\Psi_{\bar{j}}(x^*(n)) : n \in \mathbb{N})$   $w^*$ -converges in  $Z^*$  to  $\Phi_{\bar{j}}(x^*) = (x_k^*) \in \mathbb{B}^* \subset B_{Z^*}$ .

(4) In order to show that  $\mathbb{B}^*$  is  $w^*$ -compact let  $z^*(n) = (x_k^*(n)) \in \mathbb{B}^*$ , for  $n \in \mathbb{N}$ , and assume that  $(z^*(n))$  converges in  $w^*$  to some  $z^* \in Z^*$ . After passing to subsequences we can assume that there is a sequence  $\bar{j} = (j_k) \in \prod_{k=1}^{\infty} [n_k, n_{k+1}]$ , so that  $\text{rg}_E(x_k^*(n)) \subset (j_{k-1}, j_k)$  for all  $k \in \mathbb{N}$  and all  $n \geq k$ . This implies that the sequence  $(x^*(n))$ , with  $x^*(n) = \sum_{k \in \mathbb{N}} x_k^*(n)$ , for  $n \in \mathbb{N}$ , converges  $w^*$  to some element  $x^*$  which is in  $U_{\bar{j}} \cap B_{X^*}$ . It follows therefore that  $z^* = \Phi_{\bar{j}}(x^*)$ , and, thus, that  $z^* \in \mathbb{B}^*$ .

If  $z^* = (x_k^*) \in \mathbb{D}^*$  then there is some  $\bar{j} = (j_k) \in \prod_{k=1}^{\infty} [n_k, n_{k+1}]$ , so that  $x^* = \sum x_k^* \in U_{\bar{j}}$  and  $z^* = \Phi_{\bar{j}}(x^*)$ . By part (3) it follows that  $\|z^*\| = \|x^*\|$ , and since for all  $x \in X$

$$z^*(I(x)) = \sum x_k^*(P_{(n_{k-1}, n_{k+1})}^E(x)) = \sum x_k^*(x) = x^*(x),$$

it follows that  $x^* = I^*(z^*)$ . Note also that  $D^*$  consists of the union of all spaces  $U_{\bar{j}}$ , with  $\bar{j} \in \prod_{k=1}^{\infty} [n_k, n_{k+1}]$ , and that  $I^* \circ \Phi_{\bar{j}}(x^*) = x^*$  for all  $x^* \in U_{\bar{j}}$ . It follows therefore that the image of  $I^*|_{\mathbb{D}^*}$  is all of  $D^*$ .

If  $(z_i^*) \subset \mathbb{D}^*$  is a skipped block with respect to  $(Z_k^*)$ , we choose  $(m_i) \subset \mathbb{N}$  increasing so that  $\text{rg}_{Z^*}(z_i^*) \subset (m_{i-1}, m_i)$ , for  $i \in \mathbb{N}$  (with  $m_0 = 0$ ). We write for  $i \in \mathbb{N}$  the element  $z_i^*$  as  $z_i^* = (x_k^* : k = m_{i-1} + 1, m_{i-1} + 2, \dots, m_i - 1) \subset X^*$ , with  $\text{rg}_E(x_k^*) \subset (j_{k-1}, j_k)$ , for  $k = m_{i-1} + 1, m_{i-1} + 2, \dots, m_i - 1$ , where  $j_k \in [n_k, n_{k+1}]$ , for  $k = m_{i-1}, m_{i-1} + 1, \dots, m_i - 1$ . Finally we put  $x_{m_i}^* = 0$  for  $i \in \mathbb{N}$ , and deduce that  $(x_k^*) \subset U_{\bar{j}}$  with  $\bar{j} = (j_k)$ . It follows that all the  $z_i^*$ , together with their linear combinations are in the image of  $U_{\bar{j}}$  under  $\Phi_{\bar{j}}$  and it follows from the already verified facts, that  $(z_i^*)$  is isometrically equivalent to its inverse image which is the sequence  $(I^*(z_i^*))$ .

(5) We deduce from the definition (15) of the norm on  $Z$  that

$$\begin{aligned} \|T\| &= \sup_{y \in B_Y} \|T(y)\| = \sup_{y \in B_Y} \sup_{(x_k^*) \in \mathbb{B}^*} \left| \sum x_k^*(T_k(y)) \right| \\ &= \sup_{\bar{x}^* = (x_j^*) \in \mathbb{B}^*} \sup_{y \in B_Y} \left| T_{\bar{x}^*} \left( \sum x_k^* \right) (y) \right| = \sup_{\bar{x}^* \in \mathbb{B}^*} \|T_{\bar{x}^*}\| \end{aligned}$$

which proves our claim (7).  $\square$

*Remark 3.4.* Note that  $I^*|_{\mathbb{D}^*}$  is norm preserving but not injective. Indeed, let  $x^* \in B^*$  have the property that for some  $k_0 \in \mathbb{N}$  there are  $j, j' \in [n_{k_0}, n_{k_0+1}]$ , with  $j < j' - 1$ ,  $x^*|_{E_j} = x^*|_{E_{j'}} = 0$ , and there is  $i \in (j, j')$  so that  $x^*|_{E_i} \neq 0$ , then we can write  $x^* = \sum_{k=1}^{\infty} x_k^*$  and  $x^* = \sum_{k=1}^{\infty} y_k^*$ , with  $\text{rg}_E(x_{k_0}^*) \subset (0, j)$  and  $i \in \text{rg}_E(y_{k_0}^*) \not\subset (0, j)$  and thus  $(x_k^*)$  and  $(y_k^*)$  are as elements of  $Z^*$  different.

In our next step we will show that  $(Z_j)$  is shrinking in  $Z$  and we first need to recall some notion for families of finite subsets of  $\mathbb{N}$ .

**Notation.** For any set  $M$  we denote by  $[M]$ ,  $[M]^{<\omega}$  and  $[M]^\omega$  the subsets, the finite subsets, and the infinite subsets of  $M$ , respectively. If  $M = \mathbb{N}$  we introduce the following convention for subsets of  $\mathbb{N}$ . When we write  $A = \{a_1, a_2, \dots, a_n\} \in [\mathbb{N}]^{<\omega}$  or  $A = \{a_1, a_2, a_3, \dots\} \in [\mathbb{N}]^\omega$  it is implicitly assumed that the  $a_j$  are increasing.

For  $A \in [\mathbb{N}]^{<\omega}$  and  $B \in [\mathbb{N}]$  we write  $A < B$  if  $\max A < \min B$ , and introduce the convention that  $\emptyset > A$  and  $\emptyset < A$  for all  $A \in [\mathbb{N}]^{<\omega}$ . We say that  $B$  is an *extension of*  $A$ , and write  $A \preceq B$ , if  $B = A \cup B'$  for some  $B' \in [\mathbb{N}]$  with  $B' > A$ . By  $A \prec B$  we mean that  $A \neq B$  and  $A \preceq B$ .

We identify  $[\mathbb{N}]$  in the usual way with the product  $\{0, 1\}^\omega$  and consider on  $[\mathbb{N}]$  the product topology of the discrete topology on  $\{0, 1\}$ .

$\mathcal{F} \subset [\mathbb{N}]^{<\omega}$  is called *closed under restrictions* if  $A \in \mathcal{F}$  whenever  $A \prec B$  and  $B \in \mathcal{F}$ , *hereditary* if  $A \in \mathcal{F}$  whenever  $A \subset B$  and  $B \in \mathcal{F}$ , and  $\mathcal{F}$  is called *compact* if it is compact in the product topology. Note that a family which is closed under restrictions is compact if and only if it is *well founded*, i.e., if it does not contain strictly ascending chains with respect to extensions. Given  $n$ ,  $a_1 < \dots < a_n$ ,  $b_1 < \dots < b_n$  in  $\mathbb{N}$  we say that  $\{b_1, \dots, b_n\}$  is a *spread* of  $\{a_1, \dots, a_n\}$  if  $a_i \leq b_i$  for  $i = 1, \dots, n$ . A family  $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$  is called *spreading* if every spread of every element of  $\mathcal{F}$  is also in  $\mathcal{F}$ .

**Lemma 3.5.**  $(Z_j)$  is a shrinking FDD of  $Z$ .

*Proof.* Let  $(z_j)$  be a normalized block sequence in  $Z$  with respect to  $(Z_j)$ . For any  $c \in (0, 1)$  we first show that the set

$$\mathcal{A}_c = \{ \{m_1, m_2, \dots, m_l\} : \exists z^* \in \mathbb{B}^* \forall j = 1, 2, \dots, l \quad z^*(z_{m_j}) \geq c \}$$

is compact. Indeed, if our claim were not true, we could find an increasing sequence  $(m_j) \subset \mathbb{N}$  and  $z_n^* \in \mathbb{B}^*$  for each  $n \in \mathbb{N}$ , so that  $z_n^*(z_{m_j}) \geq c$  for all  $j \leq n$ . Without loss of generality we can assume that  $z_n^*$  converges in  $w^*$  to some  $z^*$  which by part (4) of Proposition 3.3 also lies in  $\mathbb{B}^*$ . We write  $z^*$  as  $z^* = (x_k^*) \subset X^*$ , and we let  $\bar{j} = (j_k) \in \prod_{k=1}^{\infty} [n_k, n_{k+1}]$ , so that  $\text{rg}_E(x_k^*) \subset (j_{k-1}, j_k)$ , for  $k \in \mathbb{N}$ .

It follows that  $\text{span}(x_k^* : k \in \mathbb{N}) \subset U_{\bar{j}}$ , and Proposition 3.3 yields that for  $i \in \mathbb{N}$

$$\left\| \sum_{k \in \text{rg}_Z(z_{m_i})} x_k^* \right\| \geq \sum_{k \in \text{rg}_Z(z_{m_i})} x_k^*(z_{m_i}) = z^*(z_{m_i}) \geq c,$$

which contradicts the convergence of the series  $\sum_{k=1}^{\infty} x_k^*$ .

We can deduce the rest of the proof from the following more generally stated result.  $\square$

**Lemma 3.6.** Let  $V$  be a Banach space having an FDD  $(V_j)$  and assume that there is a 1-norming subset  $B$  of  $B_{V^*}$  so that for some  $0 < c < 1$  and for all normalized block sequences  $(v_j)$  in  $V$  with respect to  $(V_j)$  the set

$$\mathcal{A} = \{ \{m_1, m_2, \dots, m_l\} : \exists v^* \in B \forall j = 1, 2, \dots, l \quad v^*(v_{m_j}) \geq c \}$$

is compact. Then  $(V_j)$  is shrinking in  $V$ .

Conversely if  $(V_j)$  is shrinking then for every  $0 < c < 1$  and every normalized block sequence  $(v_j)$  in  $V$  with respect to  $(V_j)$  the set

$$\mathcal{B}_c = \{ \{m_1, m_2, \dots, m_l\} : \exists v^* \in B_{V^*} \forall j = 1, 2, \dots, l \quad v^*(v_{m_j}) \geq c \}$$

is compact.

*Proof.* Assume our claim is wrong. Then by Proposition 2.1 we can choose a normalized block sequence  $(v_j)$  in  $V$  which is not weakly null. Thus, for some  $\rho \in (0, 1)$  it follows that

$$(17) \quad \left\| \sum_{j=1}^{\infty} b_j v_j \right\| \geq \rho \sum_{j=1}^{\infty} b_j, \text{ whenever } (b_i) \in c_{00}, \text{ with } b_i \geq 0, \text{ for } i \in \mathbb{N}.$$

Let  $\varepsilon = (1 - c)/3$ . Using James' argument [13] that  $\ell_1$  is not distortable, we can, by passing to further normalized blocks of the  $v_j$ , assume that  $\rho > 1 - \varepsilon$ .

Let  $\mathcal{A}$  be defined as in the statement and note that  $\mathcal{A}$  is hereditary. We recall the *Schreier space*  $X_{\mathcal{A}}$  defined for  $\mathcal{A}$ : For  $(a_i) \in c_{00}$  we put

$$\|(a_i)\|_{X_{\mathcal{A}}} = \sup_{A \in \mathcal{A}} \sum_{i \in A} |a_i|,$$

and let  $X_{\mathcal{A}}$  be the completion of  $c_{00}$  with respect to  $\|\cdot\|_{X_{\mathcal{A}}}$ . We note that the unit vector basis  $(e_i)$  is a 1-unconditional basis of  $X_{\mathcal{A}}$  and that for  $\bar{a} = (a_i) \in X_{\mathcal{A}}$  the map  $f_{\bar{a}} : \mathcal{A} \rightarrow \mathbb{R}$ ,  $B \mapsto \sum_{n \in B} a_n$ , is a continuous function defined on the countable and compact space  $\mathcal{A}$ . We compute

$$\|f_{\bar{a}}\|_{C(\mathcal{A})} = \sup_{B \in \mathcal{A}} \left| \sum_{n \in B} a_n \right| = \max \left( \sup_{B \in \mathcal{A}} \sum_{n \in B, a_n \geq 0} a_n, \sup_{B \in \mathcal{A}} \sum_{n \in B, a_n \leq 0} (-a_n) \right) \begin{cases} \geq \frac{1}{2} \|a\|_{\mathcal{A}} \\ \leq \|a\|_{\mathcal{A}}, \end{cases}$$

where the second equality follows from the fact that  $\mathcal{A}$  is hereditary. Thus,  $X_{\mathcal{A}}$  isomorphically embeds into  $C(\mathcal{A})$ , the space of continuous functions on the countable and compact space  $\mathcal{A}$ . But this means that  $\ell_1$  cannot embed into  $X_{\mathcal{A}}$ . In particular, we can find a finite sequence of non negative numbers  $(a_i)_{i=1}^l$ , with  $\sum_{i=1}^l a_i = 1$  and  $\|(a_i)\|_{X_{\mathcal{A}}} < \varepsilon$ .

Since  $\mathcal{A}$  is compact and countable  $C(\mathcal{F})$  is isometrically isomorphic to  $C[0, \alpha]$  for a countable ordinal  $\alpha$ , and thus  $X_{\mathcal{F}}$  is  $c_0$ -saturated.

Define  $v = \sum_{i=1}^l a_i v_i$  and let  $v^* \in B$ . Then  $\{i : v^*(v_i) \geq c\} \in \mathcal{A}$ , and, thus,

$$\begin{aligned} v^*(v) &= \sum_{i=1}^{\infty} a_i v^*(v_i) \\ &= \sum_{v^*(v_i) \geq c} a_i v^*(v_i) + \sum_{v^*(v_i) < c} a_i v^*(v_i) \\ &\leq \sup_{A \in \mathcal{A}} \sum_{i \in A} a_i + c \sum_{i=1}^{\infty} a_i \leq \varepsilon + c = 1 - 2\varepsilon \leq \rho - \varepsilon, \end{aligned}$$

and thus  $\|v\| = \sup_{v^* \in B} |v^*(v)| \leq \rho - \varepsilon$  which contradicts (17), and finishes the proof of our first claim.

The second claim follows from the  $w^*$ -compactness of  $B_{V^*}$  by replacing  $\mathcal{B}^*$  by  $B_{V^*}$  and arguing as in the first part of the proof of Lemma 3.5.  $\square$

**Lemma 3.7.** *If  $X$  is reflexive then  $(Z_j)$  is a boundedly complete FDD of  $Z$ .*

*Proof.* Assume that  $(Z_j)$  is not boundedly complete. Then we can find a semi normalized block sequence  $(z_j)$ , say  $\frac{1}{C} \leq \|z_i\| \leq 1$ , for all  $i \in \mathbb{N}$  and some  $C \geq 1$ , so that  $\|\sum_{j=1}^n z_j\| \leq 1$ , for all  $n \in \mathbb{N}$ . Since the set  $\mathbb{B}^*$  is 1-norming  $Z$ , we can find  $z_j^* \in \mathbb{B}^*$  so that  $z_j^*(z_j) \geq 1/2C$ . For  $i \in \mathbb{N}$ , and define  $y_i^* = P_{\text{rg}_Z(z_{2i})}^{Z^*}(z_{2i}^*)$ .

Since  $(y_i^*)$  is a semi normalized skipped block sequence with respect to  $(Z_j^*)$  it follows from Proposition 3.3 (4) that the sequence  $(I^*(y_i^*))$  ( $I : X \rightarrow Z$  as in Proposition 3.3) is a semi normalized skipped block sequence in  $D^*$  with respect to  $(E_j)$ , which is isometrically equivalent to  $(y_i^*)$ . For any sequence  $(a_j) \subset [0, 1]$  with  $\sum_{j=1}^{\infty} a_j = 1$  it follows that

$$\left\| \sum a_i I^*(y_i^*) \right\| = \left\| \sum a_i y_i^* \right\| \geq \left( \sum a_i y_i^* \right) \left( \sum_{i=1}^{2n} z_i \right) \geq \frac{1}{2C}.$$

Thus no convex block of  $(I^*(y_i^*))$  converges in norm to 0, which implies that  $(I^*(y_i^*))$  cannot converge weakly to 0, and contradicts the assumption that  $(F_j)$  is a shrinking FMD of  $X^*$ .  $\square$

From part (5) of Proposition 3.3 we also deduce the following criterium for  $(Z_j)$  being an unconditional FDD. It will depend on the choice of the FMD  $(E_j)$  of  $X$ . In Section 6

we will deduce a *coordinate free condition* on  $X$  implying that  $(Z_j)$  is unconditional, and thereby deduce the results of [16, 17].

**Proposition 3.8.** *Assume that every skipped block basis in  $X^*$  with respect to  $(F_j)$  is  $C$ -suppression unconditional. Then  $(Z_j)$  is  $C$ -suppression unconditional in  $Z$ .*

*Proof.* For each finite  $A \subset \mathbb{N}$ , we will apply (5) of Proposition 3.3 to the projection  $T = P_A^Z : Z \rightarrow Z$ , and let  $T_k : Z \rightarrow Z$ , for  $k \in \mathbb{N}$ , be defined by  $T_k = P_{\{k\}}^Z$ , if  $k \in A$ , and  $T_k = 0$ , otherwise. Since every sequence  $\bar{x}^* = (x_j^*) \in \mathbb{B}^*$  is skipped with respect to  $(F_j)$  we have for  $x^* = \sum a_k x_k^* \in \text{span}(x_j^* : j \in \mathbb{N})$  that

$$\left\| T_{\bar{x}} \left( \sum_{k=1}^{\infty} a_k x_k^* \right) \right\| = \left\| \sum_{k=1}^{\infty} a_k x_k^* \circ T_k \right\| = \left\| \sum_{k \in A} a_k x_k^* \right\| = \left\| P_A \left( \sum_{k=1}^{\infty} a_k x_k^* \right) \right\| \leq C \left\| \sum_{k=1}^{\infty} a_k x_k^* \right\|,$$

where  $P_A$  is the projection  $\overline{\text{span}(x_i^* : i \in \mathbb{N})} \rightarrow \text{span}(x_i^* : i \in A)$ ,  $\sum_{i=1}^{\infty} a_i x_i^* \mapsto \sum_{i \in A} a_i x_i^*$ , which by assumption is of norm not larger than  $C$ . This proves by part (5) of Proposition 3.3 that  $\|P_A^Z\| \leq C$ .  $\square$

Up to now we proved, that our given Banach space  $X$  with shrinking FMD  $(E_j)$  embeds into the Banach space  $Z$  which has a shrinking FDD  $(Z_j)$ . Moreover  $(Z_j)$  is boundedly complete if  $X$  is reflexive, and  $(Z_j)$  is unconditional if there is a  $C \geq 1$  so that all the skipped blocks in  $X^*$  with respect to the biorthogonal sequence  $(F_j)$  are  $C$ -unconditional.

We now show how to pass from an FDD with certain properties (shrinking, boundedly complete, and unconditional) to a basis with the same properties.

To do so assume that  $V$  is a Banach space with an FDD  $(V_j)$ . After renorming we can assume that  $(V_j)$  is bimonotone in  $V$ . We can therefore view the duals  $V_j^*$ ,  $j \in \mathbb{N}$ , to be isometrically subspaces of  $V^*$ . Moreover, in the case that the FDD  $(V_j)$  is unconditional, we also can assume, after the appropriate renorming, that it is 1-unconditional.

Let  $(\varepsilon_n) \subset (0, 1)$  be a null sequence with  $\sum \varepsilon_n < 1/3$ , and choose for each  $n \in \mathbb{N}$  a finite  $\varepsilon_n$ -net  $(x_{(n,i)}^* : i = 1, 2, \dots, l_n)$  in  $B_{V_n^*}$ . It follows that the set

$$A = \left\{ \sum a_n x_{(n,i_n)}^* : (i_n) \in \prod_{n=1}^{\infty} \{1, 2, \dots, l_n\}, \left\| \sum_{n=1}^{\infty} a_n x_{(n,i_n)}^* \right\| \leq 1 \right\}$$

is  $\frac{2}{3}$ -norming the space  $V$ . After passing to the equivalent norm defined by

$$(18) \quad \|v\| = \sup_{v^* \in A} |v^*(v)| \text{ for } v \in V$$

we can assume that  $A$  is 1-norming the space  $V$ , and also note that  $(V_j)$  is still bimonotone, respectively 1-unconditional with respect to this new norm.

We put  $\Gamma = \{(n, i) : n \in \mathbb{N}, \text{ and } i = 1, 2, \dots, l_n\}$ . We denote the unit vector basis of  $c_{00}(\Gamma) = \{(a_\gamma : \gamma \in \Gamma) \subset \mathbb{R} : \#\{\gamma : a_\gamma \neq 0\} < \infty\}$  by  $(e_\gamma : \gamma \in \Gamma)$  and its coordinate functionals by  $(e_\gamma^* : \gamma \in \Gamma)$ . We define

$$(19) \quad \begin{aligned} B &= \left\{ \sum_{n=1}^{\infty} a_n e_{(n,i_n)}^* : \sum_{n=1}^{\infty} a_n x_{(n,i_n)}^* \in A \right\} \\ &= \left\{ \sum_{n=1}^{\infty} a_n e_{(n,i_n)}^* : (i_n) \in \prod_{n=1}^{\infty} \{1, 2, \dots, l_n\}, \left\| \sum_{n=1}^{\infty} a_n x_{(n,i_n)}^* \right\| \leq 1 \right\}. \end{aligned}$$

Then we define on  $c_{00}(\Gamma)$  the norm

$$(20) \quad \|x\| = \sup_{w^* \in B} w^*(x), \text{ for } x \in c_{00}(\Gamma).$$

Let  $W$  be the completion of  $c_{00}(\Gamma)$  with respect to  $\|\cdot\|$ .

**Theorem 3.9.** *Let  $V$  and  $W$  be as introduced above. Then  $(e_\gamma : \gamma \in \Gamma)$  is a basis of  $W$ , where the set  $\Gamma$  is lexicographically ordered. The map*

$$J : V \rightarrow W, \quad \sum v_n \mapsto \sum_{n=1}^{\infty} \sum_{i=1}^{l_n} x_{(n,i)}^*(v_n) e_{(n,i)}.$$

*is an isometric embedding of  $V$  into  $W$  (using the norm in (18)), and*

- a)  $(e_\gamma : \gamma \in \Gamma)$  is 1 unconditional, if  $(V_j)$  is 1-unconditional,
- b)  $(e_\gamma : \gamma \in \Gamma)$  is shrinking, if  $(V_j)$  is shrinking, and
- c)  $(e_\gamma : \gamma \in \Gamma)$  is shrinking and boundedly complete if  $V$  is reflexive.

*Remark 3.10.* The construction of  $W$ , appears already in [19, Theorem 1.g.5], where it was shown that  $(e_\gamma : \gamma \in \Gamma)$  is unconditional, if  $(V_j)$  is unconditional. In [19] the space  $W$  is defined by its unit ball, not by its norming set  $B$ . It was already mentioned in [17] that this construction leads to a shrinking basis in the case that  $(V_j)$  is a shrinking FDD. Nevertheless we will, to be self-contained, present the complete argument, and later we will show that the space  $W$  has the same Szlenk index as the space  $V$ , and that, in the case that  $V$  is reflexive, also  $W^*$  and  $V^*$  share the same Szlenk index.

*Proof.* First we prove that  $(e_\gamma : \gamma \in \Gamma)$ , ordered lexicographically, is bimonotone. Indeed, denote the lexicographical order on  $\Gamma$  by  $\preceq$ . For  $\gamma_- = (m, j_-)$  and  $\gamma_+ = (n, j_+)$  in  $\Gamma$ , with  $m \leq n$  and  $j_- < j_+$ , if  $m = n$ , and  $w^* = \sum_{k=1}^{\infty} a_k e_{(k, i_k)}^* \in B$  it follows from the bimonotonicity of  $(V_j)$  that

$$P_{[\gamma_-, \gamma_+]}^*(w^*) := \sum_{\gamma_- \preceq (k, i_k) \preceq \gamma_+} a_k e_{(k, i_k)}^* = \begin{cases} \sum_{k=m}^n a_k e_{(k, i_k)}^* & \text{if } j_- \leq i_m \text{ and } i_n \leq j_+, \\ \sum_{k=m+1}^n a_k e_{(k, i_k)}^* & \text{if } j_- > i_m \text{ and } i_n \leq j_+, \\ \sum_{k=m}^{n-1} a_k e_{(k, i_k)}^* & \text{if } j_- \leq i_m \text{ and } i_n > j_+, \\ \sum_{k=m+1}^{n-1} a_k e_{(k, i_k)}^* & \text{if } j_- > i_m \text{ and } i_n > j_+, \end{cases}$$

and since the set  $A$  is closed under projections of the form  $P_{[i, j]}^V$  it follows that  $P_{[\gamma_-, \gamma_+]}(w^*) \in B$ . This yields for  $w = \sum \xi_\gamma e_\gamma \in c_{00}(\Gamma)$  that

$$\left\| P_{[\gamma_-, \gamma_+]} \left( \sum \xi_\gamma e_\gamma \right) \right\| = \left\| \sum_{\gamma_- \preceq \gamma \preceq \gamma_+} \xi_\gamma e_\gamma \right\| = \sup_{w^* \in B} P_{[\gamma_-, \gamma_+]}^*(w^*)(w) \leq \|w\|.$$

For  $v = \sum_{n=1}^{\infty} v_n \in V$ , with  $v_n \in V_n$ , for  $n \in \mathbb{N}$ , we have

$$\begin{aligned} \left\| J \left( \sum_{n=1}^{\infty} v_n \right) \right\| &= \sup \left\{ \left\| \sum_{n=1}^{\infty} a_n x_{(n, i_n)}^*(v_n) : (i_n) \in \prod_{n=1}^{\infty} \{1, 2, \dots, l_n\}, \left\| \sum_{n=1}^{\infty} a_n x_{(n, i_n)}^* \right\| \leq 1 \right\} \\ &= \sup \left\{ \left\| \left( \sum_{n=1}^{\infty} a_n x_{(n, i_n)}^* \right) \left( \sum_{n=1}^{\infty} v_n \right) : \sum_{n=1}^{\infty} a_n x_{(n, i_n)}^* \in A \right\} \\ &= \left\| \sum_{n \in \mathbb{N}} v_n \right\|, \end{aligned}$$



and thus  $J$  is an isometric embedding of  $V$  into  $W$ . Assume that  $(V_j)$  is 1-unconditional. In order to show that  $(e_\gamma : \gamma \in \Gamma)$  is 1-unconditional we observe for  $(\xi_\gamma : \gamma \in \Gamma) \in c_{00}(\Gamma)$  and  $(\sigma_\gamma : \gamma \in \Gamma) \in \{-1, 1\}^\Gamma$  that

$$\begin{aligned} \left\| \sum_{\gamma \in \Gamma} \sigma_\gamma \xi_\gamma e_\gamma \right\| &= \sup \left\{ \left\| \sum_{n=1}^{\infty} a_n \sigma_{(n, i_n)} \xi_{(n, i_n)} : (i_n) \in \prod_{n=1}^{\infty} \{1, 2, \dots, l_n\}, \left\| \sum_{n=1}^{\infty} a_n x_{(n, i_n)}^* \right\| \leq 1 \right\} \\ &= \sup \left\{ \left\| \sum_{n=1}^{\infty} a_n \xi_{(n, i_n)} : (i_n) \in \prod_{n=1}^{\infty} \{1, 2, \dots, l_n\}, \left\| \sum_{n=1}^{\infty} a_n x_{(n, i_n)}^* \right\| \leq 1 \right\} = \left\| \sum_{\gamma \in \Gamma} \xi_\gamma e_\gamma \right\|, \end{aligned}$$

where the second equality follows from the equivalence

$$\left\| \sum_{n=1}^{\infty} a_n x_{(n, i_n)}^* \right\| \leq 1 \iff \left\| \sum_{n=1}^{\infty} a_n \sigma_{(n, i_n)} x_{(n, i_n)}^* \right\| \leq 1.$$

For  $n \in \mathbb{N}$  put  $W_n = \text{span}(e_{(n, j)} : j = 1, 2, \dots, l_n)$ , and note that  $(W_n)$  is an FDD of  $W$ . Let  $(w_m^*) \subset B$  be a normalized block with respect to  $(W_n^*)$ . For  $m \in \mathbb{N}$  we write  $w_m^* = \sum_{j=k_{m-1}+1}^{k_m} a_j x_{(j, i_j)}^*$ , for some sequences  $(i_j) \in \prod_{j=1}^{\infty} \{1, 2, \dots, l_j\}$ ,  $k_1 < k_2 < \dots$  in  $\mathbb{N}$ , and  $(a_j) \subset \mathbb{R}$ . For  $m \in \mathbb{N}$  define  $v_m^* = w_m^*|_V \in V^*$ . On the one hand it follows for any  $(b_m) \subset c_{00}$ , that  $\left\| \sum_{m=1}^{\infty} b_m w_m^* \right\| \geq \left\| \sum_{m=1}^{\infty} b_m v_m^* \right\|$ . On the other hand, if  $\left\| \sum_{m=1}^{\infty} b_m v_m^* \right\| = \left\| \sum_{m=1}^{\infty} \sum_{j=k_{m-1}+1}^{k_m} a_j x_{(j, i_j)}^* \right\| = 1$ , then  $\sum_{m=1}^{\infty} b_m v_m^* \in A$ , and thus  $\sum_{m=1}^{\infty} b_m w_m^* \in B$ , which, by definition of the norm on  $W$  means that  $\left\| \sum_{m=1}^{\infty} b_m w_m^* \right\| \leq 1$ . We thus proved that the sequences  $(w_m^*)$  and  $(w_m^*|_V)$  are isometrically equivalent.

Assume now that  $(V_j)$  is shrinking. To show that  $(e_\gamma : \gamma \in \Gamma)$  is shrinking, it will be enough to show that  $(W_n)$  is a shrinking FDD of  $W$ . Assume that this were not true, and that by Lemma 3.6 there is a  $0 < c < 1$ , a normalized block  $(w_j)$  in  $W$  with respect to  $(W_j)$ , an increasing sequence  $(m_j)$  and for each  $n \in \mathbb{N}$  an element  $w^*(n) \in B$ , so that  $w^*(n)(w_{m_j}) \geq c$ , for all  $j = 1, 2, \dots, n$ . After passing to a subsequence we can assume  $w^* = w^* - \lim_{n \rightarrow \infty} w^*(n)$  exists. Put  $w_k^* = P_{(\max \text{rg}_W(w_{m_{k-1}}), \max \text{rg}_W(w_{m_k}))}^{W^*}(w^*)$ , for  $k \in \mathbb{N}$ .

It follows that  $\|w_k^*\| \geq |w_k^*(w_{m_k})| \geq c$ , and that  $\left\| \sum_{k=1}^n w_k^* \right\| = \left\| P_{[1, \max \text{rg}_W(w_{m_k})]}^{W^*}(w^*) \right\| \leq 1$ . Thus, the previously observed equivalence between  $(w_k^*)$  and  $(w_k^*|_V)$  yields that  $\|w_k^*|_V\| \geq c$ , for  $k \in \mathbb{N}$ , and  $\left\| \sum_{k=1}^n w_k^*|_V \right\| \leq 1$ , for  $n \in \mathbb{N}$ , which contradicts the assumption that  $(V_j)$  is shrinking in  $V$  and thus that  $(V_j^*)$  is boundedly complete in  $V^*$ .

Finally assume that  $V$  is reflexive. Again we only need to show that  $(W_j)$  is boundedly complete. Assume that  $(W_n)$  is not boundedly complete and that we can find a normalized block  $(w_j)$  in  $W$  with respect to  $(W_j)$  so that  $C = \sup_{n \in \mathbb{N}} \left\| \sum_{j=1}^n w_j \right\| < \infty$ . For each  $n \in \mathbb{N}$  we choose  $w_n^* \in B$  so that  $w_n^*(w_n) \geq 1/2$ . Since  $(W_n)$  is bimonotone, we can assume that also  $(w_n^*)$  is a block sequence in  $W^*$  with respect to  $(W_j^*)$ , and therefore, we deduce from the isometric equivalence between  $(w_j^*)$  and  $(w_j^*|_V)$ , that

$$\left\| \sum_{j=1}^n w_j^*|_V \right\| \geq \left\| \sum_{j=1}^n w_j^* \right\| \geq \frac{1}{C} \left( \sum_{j=1}^n w_j^* \right) \left( \sum_{j=1}^n w_j \right) \geq \frac{n}{2C}.$$

which is a contradiction to the assumption that  $V$  is reflexive.  $\square$

## 4. ORDINAL INDICES FOR TREES

The aim of this section is to introduce certain ordinal indices of trees, and prove some results which will later be needed to compute the Szlenk indices of the spaces  $Z$  and  $W$ , as defined in Section 3. We first follow the exposition of [24] and recall some of the notation introduced there. We begin with defining a general class of ordinal indices of trees on arbitrary sets.

Let  $M$  be an arbitrary set. We set  $M^{<\omega} = \bigcup_{n=0}^{\infty} M^n$ , the set of all finite sequences in  $M$ , which includes the sequence of length zero which is  $\emptyset$ . For  $x \in M$  we shall write  $x$  instead of  $(x)$ , *i.e.*, we identify  $M$  with sequences of length 1 in  $M$ . A *tree on  $M$*  is a non-empty subset  $\mathcal{A}$  of  $M^{<\omega}$  closed under taking initial segments: if  $(x_1, \dots, x_n) \in \mathcal{A}$  and  $0 \leq m \leq n$ , then  $(x_1, \dots, x_m) \in \mathcal{A}$ . A tree  $\mathcal{A}$  on  $M$  is *hereditary* if every subsequence of every member of  $\mathcal{A}$  is also in  $\mathcal{A}$ .

Given  $\bar{x} = (x_1, \dots, x_m)$  and  $\bar{y} = (y_1, \dots, y_n)$  in  $M^{<\omega}$ , we write  $(\bar{x}, \bar{y})$  for the concatenation of  $\bar{x}$  and  $\bar{y}$ :

$$(\bar{x}, \bar{y}) = (x_1, \dots, x_m, y_1, \dots, y_n).$$

Given  $\mathcal{A} \subset M^{<\omega}$  and  $\bar{x} \in M^{<\omega}$ , we let

$$\mathcal{A}(\bar{x}) = \{\bar{y} \in M^{<\omega} : (\bar{x}, \bar{y}) \in \mathcal{A}\}.$$

Note that if  $\mathcal{A}$  is a tree on  $M$ , then so is  $\mathcal{A}(\bar{x})$  (unless it is empty). Moreover, if  $\mathcal{A}$  is hereditary, then so is  $\mathcal{A}(\bar{x})$  and  $\mathcal{A}(\bar{x}) \subset \mathcal{A}$ .

Let  $M^\omega$  denote the set of all (infinite) sequences in  $M$ . Fix a set of  $M$ -valued sequences  $S \subset M^\omega$ . For a tree  $\mathcal{A}$  on  $M$  the  *$S$ -derivative*  $\mathcal{A}'_S$  of  $\mathcal{A}$  consists of all finite sequences  $\bar{x} \in M^{<\omega}$  for which there is a sequence  $(y_i)_{i=1}^\infty \in S$  with  $(\bar{x}, y_i) \in \mathcal{A}$  for all  $i \in \mathbb{N}$ . Note that  $\mathcal{A}'_S \subset \mathcal{A}$ , but that in general  $\mathcal{A}'_S$  does not need to be a tree. Nevertheless if we assume that  $\mathcal{A}$  is hereditary, then  $\mathcal{A}'_S$  is also a hereditary tree (unless it is empty). We then define higher order derivatives  $\mathcal{A}_S^{(\alpha)}$  for ordinals  $\alpha < \omega_1$  by recursion as follows.

$$\mathcal{A}_S^{(0)} = \mathcal{A}, \quad \mathcal{A}_S^{(\alpha+1)} = (\mathcal{A}_S^{(\alpha)})'_S, \quad \text{for } \alpha < \omega_1 \quad \text{and} \quad \mathcal{A}_S^{(\lambda)} = \bigcap_{\alpha < \lambda} \mathcal{A}_S^{(\alpha)} \quad \text{for a limit ordinal } \lambda < \omega_1.$$

It is clear that  $\mathcal{A}_S^{(\alpha)} \supset \mathcal{A}_S^{(\beta)}$ , whenever  $\alpha \leq \beta$ , and if  $\mathcal{A}$  is a hereditary tree it follows that  $\mathcal{A}_S^{(\alpha)}$  is also a hereditary tree (or the empty set). An easy induction also shows that

$$(21) \quad (\mathcal{A}(\bar{x}))_S^{(\alpha)} = (\mathcal{A}_S^{(\alpha)})(\bar{x}) \quad \text{for all } \bar{x} \in M^{<\omega}, \alpha < \omega_1.$$

We now define the  *$S$ -index*  $I_S(\mathcal{A})$  of  $\mathcal{A}$  by

$$I_S(\mathcal{A}) = \min\{\alpha < \omega_1 : \mathcal{A}_S^{(\alpha)} = \emptyset\}$$

if there exists  $\alpha < \omega_1$  with  $\mathcal{A}_S^{(\alpha)} = \emptyset$ , and  $I_S(\mathcal{A}) = \omega_1$  otherwise.

We note for  $\bar{x} \in M^{<\omega}$ , an hereditary tree  $\mathcal{A} \subset [M]^\omega$  and  $\alpha < \omega_1$  that

$$(22) \quad I_S(\mathcal{A}(\bar{x})) \geq \alpha + 1 \iff \emptyset \in \mathcal{A}^{(\alpha)}(\bar{x}) \iff \bar{x} \in \mathcal{A}^{(\alpha)} \text{ and}$$

$$(23) \quad I_S(\mathcal{A}(\bar{x})) \geq \alpha + 2 \iff \exists (y_j) \in S \forall j \in \mathbb{N} \quad I_S(\mathcal{A}(\bar{x}, y_j)) \geq \alpha + 1.$$

*Remark 4.1.* If  $\lambda$  is a limit ordinal and  $\mathcal{A}_S^{(\alpha)} \neq \emptyset$  for all  $\alpha < \lambda$ , then in particular  $\emptyset \in \mathcal{A}_S^{(\alpha)}$  for all  $\alpha < \lambda$ , and hence  $\mathcal{A}_S^{(\lambda)} = \bigcap_{\alpha < \lambda} \mathcal{A}_S^{(\alpha)} \neq \emptyset$ . This shows that  $I_S(\mathcal{A})$  is always a successor ordinal.

**Examples 4.2.** A tree  $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$  (see the notation introduced in Section 3) can be thought of as a tree on  $\mathbb{N}$ : a set  $F = \{m_1, \dots, m_k\} \in [\mathbb{N}]^{<\omega}$ , with  $m_1 < m_2 < \dots < m_k$ , is identified with the increasing sequence  $(m_1, \dots, m_k) \in \mathbb{N}^{<\omega}$ . Let  $S$  be the set of all strictly increasing sequences in  $\mathbb{N}$ . In this case the  $S$ -index of a hereditary tree  $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$  is nothing else but the *Cantor-Bendixson index* which we denote by  $\text{CB}(\mathcal{F})$  of  $\mathcal{F}$ . For the derivative, or more generally, the  $\alpha$ -derivative of  $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ , with respect to  $S$ , we will use  $\mathcal{F}'$  and  $\mathcal{F}^{(\alpha)}$ , instead of  $\mathcal{F}'_S$  and  $\mathcal{F}_S^{(\alpha)}$ . Recall that the *Cantor-Bendixson derivative* of  $\mathcal{F}$  for a hereditary tree  $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$ , is

$$\mathcal{F}' = \mathcal{F}'_{[\mathbb{N}]^\omega} = \left\{ \{a_1, a_2, \dots, a_l\} : \begin{array}{l} \exists \{n_j : j \in \mathbb{N}\} \subset [\{a_l + 1, a_l + 2, \dots\}]^\omega \\ \{a_1, a_2, \dots, a_l, n_j\} \in \mathcal{F}, \text{ for all } j \in \mathbb{N} \end{array} \right\},$$

Note that if  $\mathcal{F}$  is compact, then  $\mathcal{F}'$  is compact, and  $\mathcal{F}' \subset \mathcal{F}$ . As already noted in Section 3, if  $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$  is closed under restrictions, then  $\mathcal{F}$  is compact if and only if it is *well founded*, i.e., does not contain an infinite chain, and thus every  $A \in \mathcal{F}$  can be extended to a maximal element in  $\mathcal{F}$ . We denote the maximal elements of  $\mathcal{F}$  by  $\text{MAX}(\mathcal{F})$ . Since  $[\mathbb{N}]^{<\omega}$  is a Polish space, we deduce that the Cantor-Bendixson index  $\text{CB}(\mathcal{F})$ , of a hereditary tree  $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$  is countable if and only if  $\mathcal{F}$  is compact.

If  $M$  is an arbitrary set and  $S = M^\omega$  (which includes the constant sequences), then the  $S$ -index of a hereditary tree  $\mathcal{A}$  on  $M$  is what is usually called *the order of  $\mathcal{A}$*  (or *the height of  $\mathcal{A}$* ) denoted by  $\text{o}(\mathcal{A})$ . Note that in this case the  $S$ -derivative of  $\mathcal{A}$  consists of all non maximal elements of  $\mathcal{A}$ . The function  $\text{o}(\cdot)$  is the largest index: for any  $S \subset X^\omega$  we have  $\text{o}(\mathcal{A}) \geq \text{I}_S(\mathcal{A})$ .

We say that  $S \subset X^\omega$  *contains diagonals* if every subsequence of every member of  $S$  also belongs to  $S$  and for every sequence  $(x_n)$  in  $S$  with  $x_n = (x_{(n,i)})_{i=1}^\infty$  there exist  $i_1 < i_2 < \dots$  in  $\mathbb{N}$  such that  $(x_{(n,i_n)})_{n=1}^\infty$  belongs to  $S$ .

One way to compute ordinal indices of hereditary trees on general sets, is to find order isomorphisms between them and the *Schreier Sets*  $\mathcal{S}_\alpha$  and the *Fine Schreier Sets*  $\mathcal{F}_\alpha$ , for  $\alpha < \omega_1$ , which we want to recall now. We first fix for every limit ordinal  $\alpha < \omega_1$  a sequence  $(\lambda(\alpha, n))_{n \in \mathbb{N}}$  of ordinals with  $1 \leq \lambda(\alpha, n) \nearrow \alpha$ . We want to make sure that  $\mathcal{F}_{\omega^\alpha} = \mathcal{S}_\alpha$ , for all  $\alpha < \omega_1$ , and therefore need to make a very specific choice for  $(\lambda(\alpha, n))_{n \in \mathbb{N}}$  which we define by transfinite induction for all limit ordinals  $\alpha$ . If  $\alpha = \omega$  we put  $\lambda(\alpha, n) = n$  and assuming that  $(\lambda(\gamma, n))_{n \in \mathbb{N}}$  has been defined for all limit ordinals  $\gamma < \alpha$ , we first write  $\alpha$  in its *Cantor Normal Form* which for a limit ordinal has the (uniquely defined) form

$$\alpha = \omega^{\xi_l} k_l + \omega^{\xi_{l-1}} k_{l-1} + \dots + \omega^{\xi_1} k_1$$

with  $l \in \mathbb{N}$ ,  $\xi_l > \xi_{l-1} > \dots > \xi_1 \geq 1$  and  $k_1, k_2, \dots, k_l \in \mathbb{N}$  and put

$$\lambda(\alpha, n) = \begin{cases} \omega^{\xi_l} + \lambda(\omega^{\xi_{l-1}} k_{l-1} + \omega^{\xi_{l-2}} k_{l-2} + \dots + \omega^{\xi_1} k_1, n) & \text{if } l \geq 2, \\ \omega^{\xi_l} n & \text{if } l=1 \text{ and } \xi_l = \zeta + 1, \\ \omega^{\lambda(\xi_l, n)} & \text{if } l=1, \xi_l \text{ is limit ordinal, and } \xi_l < \omega^{\xi_l}, \\ \omega^{\beta_n} & \text{if } l=1, \xi_l \text{ is limit ordinal, and } \xi_l = \omega^{\xi_l}, \end{cases}$$

where in the fourth case we choose an arbitrary but fixed sequence  $(\beta_n) \subset [0, \xi_l)$  which increases to  $\xi_l$ .

We define the *fine Schreier families*  $(\mathcal{F}_\alpha)_{\alpha < \omega_1}$  by recursion:

$$\begin{aligned} \mathcal{F}_0 &= \{\emptyset\}, \quad \mathcal{F}_{\alpha+1} = \{\{n\} \cup A : n \in \mathbb{N}, A \in \mathcal{F}_\alpha\} \cup \{\emptyset\} \\ \mathcal{F}_\alpha &= \{A \in [\mathbb{N}]^{<\omega} : \exists n \leq \min A, A \in \mathcal{F}_{\lambda(\alpha, n)}\}, \text{ if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

An easy induction shows that  $\mathcal{F}_\alpha$  is a hereditary, compact and spreading family for all  $\alpha < \omega_1$ . Moreover,  $(\mathcal{F}_\alpha)_{\alpha < \omega_1}$  is an “almost” increasing chain:

$$(24) \quad \forall \alpha \leq \beta < \omega_1 \quad \exists n \in \mathbb{N} \quad \forall F \in \mathcal{F}_\alpha \quad \text{if } n \leq \min F, \text{ then } F \in \mathcal{F}_\beta.$$

This can also be proved by an easy induction on  $\beta$ . We note also that for  $A \in \mathcal{F}_\alpha \setminus \text{MAX}(\mathcal{F}_\alpha)$  we have  $A \cup \{n\} \in \mathcal{F}_\alpha$  for all  $n \geq \max A$ . Using transfinite induction it follows for the Cantor Bendixson index of the fine Schreier families that  $\text{CB}(\mathcal{F}_\alpha) = \alpha + 1$  for all  $\alpha < \omega_1$ . The fact that  $\mathcal{F}_\alpha$  is spreading implies moreover that

$$(25) \quad \text{CB}(\mathcal{F}_\alpha \cap [N]^{<\omega}) = \alpha + 1, \text{ for all } \alpha < \omega_1 \text{ and all } N \in [\mathbb{N}]^\omega.$$

We define the *Schreier family of order  $\alpha$*  by  $\mathcal{S}_\alpha = \mathcal{F}_{\omega^\alpha}$  for all  $\alpha < \omega_1$ . This is not exactly how the Schreier families are usually defined, but thanks to our special choice of the sequence  $(\lambda(\alpha, n))_{n \in \mathbb{N}}$  for limit ordinals both definitions coincide as noted in the following proposition. We will also put  $\mathcal{S}_{\alpha, n} = \mathcal{F}_{\omega^\alpha \cdot n}$ , for  $\alpha < \omega$  and  $n \in \mathbb{N}$ .

**Proposition 4.3.** *Let  $\alpha < \omega_1$  and  $n \in \mathbb{N}$ .*

$$(26) \quad \mathcal{S}_{\alpha, n} = \left\{ \bigcup_{j=1}^n E_j : E_j \in \mathcal{S}_\alpha, j = 1, 2, \dots, n, \text{ and } E_1 < E_2 < \dots < E_n \right\}$$

$$(27) \quad \mathcal{S}_\alpha = \left\{ \bigcup_{j=1}^n E_j : n \leq \min(E_1), E_1 < E_2 < \dots < E_n, E_j \in \mathcal{S}_\beta, j = 1, 2, \dots, \right\} \text{ if } \alpha \leq \beta + 1$$

$$(28) \quad \mathcal{S}_\alpha = \{E : \exists k \leq \min(E), \text{ with } E \in \mathcal{S}_{\lambda(\alpha, k)}\} \text{ if } \alpha \text{ is a limit ordinal.}$$

*Sketch.* We first prove the following claim by transfinite induction for all  $\alpha < \omega_1$ .

**Claim.** Assume the Cantor normal form of  $\alpha$  is

$$\alpha = \omega^{\xi_l} k_l + \omega^{\xi_{l-1}} k_{l-1} + \dots + \omega^{\xi_1} k_1$$

with  $l \in \mathbb{N}$ ,  $\xi_l > \xi_{l-1} > \dots > \xi_1 \geq 0$  and  $k_l, k_{l-1}, \dots, k_1 \in \mathbb{N}$ . Then for all ordinals  $\beta$  of the form

$$\beta = \omega^{\xi_{l+m}} k_{l+m} + \omega^{\xi_{l+m-1}} k_{l+m-1} + \dots + \omega^{\xi_{l+1}} k_{l+1},$$

with  $m \in \mathbb{N}$ ,  $\xi_{l+m} > \xi_{l+m-1} > \dots > \xi_{l+1} \geq \xi_l$  and  $k_{l+m}, k_{l+m-1}, \dots, k_{l+1} \in \mathbb{N}$ , it follows that

$$\mathcal{F}_{\beta+\alpha} = \mathcal{F}_\alpha \sqcup_{<} \mathcal{F}_\beta := \{E \cup F : E \in \mathcal{F}_\alpha, F \in \mathcal{F}_\beta, E < F\}.$$

Using the claim we can prove (26) by induction for all  $n \in \mathbb{N}$ . Then (27) and (28) follow by transfinite induction, where in the induction step (27) follows from (26), the definition of  $\mathcal{S}_\alpha$  and the choice of  $(\lambda(\omega^\alpha, n) : n \in \mathbb{N})$  if  $\alpha$  is a successor ordinal, and (28) follows from the definition of  $\mathcal{S}_\alpha$  and the choice of  $(\lambda(\omega^\alpha, n) : n \in \mathbb{N})$  if  $\alpha$  is a limit ordinal.  $\square$

For our next observation we need the following notation. Given a family  $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$  on  $\mathbb{N}$ , and a family  $(x_F)_{F \in \mathcal{F} \setminus \{\emptyset\}}$  in a set  $M$ , indexed by  $\mathcal{F}$ ,  $(\bar{x}_F)_{F \in \mathcal{F}}$  denotes the set of corresponding branches, i.e.  $\bar{x}_\emptyset = \emptyset$  and for  $F = \{m_1, m_2, \dots, m_l\} \in \mathcal{F} \setminus \{\emptyset\}$  we let

$$\bar{x}_F = (x_{\{m_1\}}, x_{\{m_1, m_2\}}, \dots, x_{\{m_1, m_2, \dots, m_l\}}).$$

**Proposition 4.4.** [24, Proposition 5] *Let  $M$  be a set and assume that  $S \subset M^\omega$  contains diagonals. Then for a hereditary tree  $\mathcal{A}$  on  $M$  and  $\alpha < \omega_1$  the following are equivalent.*

- (i)  $\alpha < \text{I}_S(\mathcal{A})$ .
- (ii) *There is a family  $(x_F)_{F \in \mathcal{F}_\alpha \setminus \{\emptyset\}} \subset M$  such that  $(\bar{x}_F)_{F \in \mathcal{F}_\alpha} \subset \mathcal{A}$  and for all  $F \in \mathcal{F}_\alpha \setminus \text{MAX}(\mathcal{F}_\alpha)$  the sequence  $(x_{F \cup \{n\}})_{n > \max F}$  is in  $S$ .*

*Remark 4.5.* Let  $\alpha < \omega_1$  and  $\mathcal{A} \subset M^{<\omega}$  be a hereditary tree. Assume that the family  $(x_F)_{F \in \mathcal{F}_\alpha \setminus \{\emptyset\}} \subset M$  satisfies the conditions in (ii) of Proposition 4.4. Then the map

$$\pi : \mathcal{F}_\alpha \rightarrow \mathcal{A}, \quad \pi(\emptyset) = \emptyset, \quad \pi(F) = \bar{x}_F \text{ if } F \in \mathcal{F}_\alpha \setminus \{\emptyset\}$$

is an order isomorphism from  $\mathcal{F}_\alpha$  to  $\mathcal{A}$ , such that that  $\pi(F \cup \{n\}) = (\pi(F), \{x_{F \cup \{n\}}\})$ , if  $n > \max(F)$ , and  $(x_{F \cup \{n\}} : n > \max(F)) \in S$  whenever  $F \in \mathcal{F}_\alpha \setminus \text{MAX}(\mathcal{F}_\alpha)$ .

In the case of  $M = \mathbb{N}$  and  $S = [\mathbb{N}]^\omega$  (see Examples 4.2) we deduce therefore that if  $\mathcal{A} \subset [\mathbb{N}]^{<\omega}$  is hereditary and compact, then  $\text{CB}(\mathcal{A}) > \alpha$  if and only if there is an order isomorphism  $\pi : \mathcal{F}_\alpha \rightarrow \mathcal{A}$ , so that for all  $A \in \mathcal{F}_\alpha \setminus \text{MAX}(\mathcal{F}_\alpha)$  and  $n > \max(A)$  it follows that  $\pi(A \cup \{n\}) = \pi(A) \cup \{s_n\}$ , where  $(s_n)$  is an increasing sequence in  $\{s \in \mathbb{N} : s > \max \pi(A)\}$ .

**Example 4.6.** *The weak index.* Let  $X$  be a separable Banach space. Let  $S$  be the set of all weakly null sequences in  $S_X$ , the unit sphere of  $X$ . We call the  $S$ -index of a hereditary tree  $\mathcal{F}$  on  $S_X$  the *weak index of  $\mathcal{F}$*  and we shall denote it by  $I_w(\mathcal{F})$ . We shall use the term *weak derivative* instead of  $S$ -derivative and use the notation  $\mathcal{F}'_w$  and  $\mathcal{F}_w^{(\alpha)}$ . When the dual space  $X^*$  is separable, the weak topology on the unit ball  $B_X$  of  $X$  is metrizable. Hence in this case the set  $S$  contains diagonals and Proposition 4.4 applies.

We now recall two important results on Schreier families. The first one can be found in [10] and is an application of Ramsey's Theorem.

**Lemma 4.7.** [10, Theorem 1.1] *Assume that  $\mathcal{F}, \mathcal{G} \subset [\mathbb{N}]^{<\omega}$  are two hereditary families and  $M \in [\mathbb{N}]^\omega$ . Then there is an  $N \in [M]^\omega$  so that  $\mathcal{F} \cap [N]^{<\omega} \subset \mathcal{G}$  or  $\mathcal{G} \cap [N]^{<\omega} \subset \mathcal{F}$ . In particular, if  $\text{CB}(\mathcal{F} \cap [M]^{<\omega}) < \text{CB}(\mathcal{G} \cap [M]^{<\omega})$ , for all  $M \in [\mathbb{N}]^\omega$ , then the second alternative cannot happen, and thus, for all  $M \in [\mathbb{N}]^\omega$  there is an  $N \in [M]^\omega$  so that  $\mathcal{F} \cap [N]^{<\omega} \subset \mathcal{G}$ .*

In order to state the next result due to Argyros and Gasparis [2] we will need further notation and the following observation, which can be easily shown by transfinite induction.

**Lemma 4.8.** *Let  $\alpha < \omega_1$  then*

- (1)  $A \in \text{MAX}(\mathcal{S}_{\alpha+1})$  if and only if  $A = \bigcup_{j=1}^n A_j$ , with  $n = \min(A_1)$  and  $A_1 < A_2 < \dots < A_n$  are in  $\text{MAX}(\mathcal{S}_\alpha)$ . In this case the sets  $A_j$ ,  $j = 1, 2, \dots, n$  are unique.
- (2) If  $\alpha$  is a limit ordinal then  $A \in \text{MAX}(\mathcal{S}_\alpha)$  if and only if there exists an  $n \leq \min(A)$  so that  $A \in \text{MAX}(\mathcal{S}_{\lambda(\alpha, n)})$  and for all  $k \in \mathbb{N}$ ,  $k > \max(A)$  it follows that  $A \cup \{k\} \notin \bigcup_{j=1}^{\min(A)} \mathcal{S}_{\lambda(\alpha, j)}$ .

For each  $\alpha < \omega_1$  and each  $A \in \text{MAX}(\mathcal{S}_\alpha)$  we will introduce a probability measure  $\mathbb{P}_{(\alpha, A)}$  on  $\mathbb{N}$  whose support is  $A$ . If  $\alpha = 0$  then  $\mathcal{S}_0 = \mathcal{F}_1$  consists of singletons and for  $A = \{n\} \in \mathcal{S}_0$  we put  $\mathbb{P}_{(0, \{n\})} = \delta_n$ , the Dirac measure in  $n$ . Assume for all  $\gamma < \alpha$  and all  $A \in \text{MAX}(\mathcal{S}_\gamma)$  we already have introduced  $\mathbb{P}_{(\gamma, A)}$  which we write as  $\mathbb{P}_{(\gamma, A)} = \sum_{a \in A} p_{(\gamma, A)}(a) \delta_a$ , with  $p_{(\gamma, A)} > 0$  for all  $a \in A$ . If  $\alpha = \gamma + 1$  for some  $\gamma < \omega_1$  and if  $A \in \text{MAX}(\mathcal{S}_\alpha)$  we write by Lemma 4.8 (1)  $A$  in a unique way as  $A = \bigcup_{j=1}^n A_j$ , with  $n = \min A$  and  $A_1 < A_2 < \dots < A_n$  are maximal in  $\mathcal{S}_\gamma$ . We then define

$$\mathbb{P}_{(\alpha, A)} = \frac{1}{n} \sum_{j=1}^n \mathbb{P}_{(\gamma, A_j)} = \frac{1}{n} \sum_{j=1}^n \sum_{a \in A_j} p_{(\gamma, A_j)}(a) \delta_a,$$

and thus

$$p_{(\alpha, A)}(a) = \frac{1}{n} p_{(\gamma, A_j)}(a) \text{ for } j = 1, 2, \dots, n \text{ and } a \in A_j.$$

If  $\alpha$  is a limit ordinal and  $A \in \text{MAX}(\mathcal{S}_\alpha)$  then

$$m = \min\{n \leq \min(A) : A \in \mathcal{S}_{\lambda(\alpha, n)}\}$$

exists and by Lemma 4.8 (2) we have that  $A \in \text{MAX}(\mathcal{S}_{\lambda(\alpha, m)})$  and can therefore put

$$\mathbb{P}_{(\alpha, A)} = \mathbb{P}_{(\lambda(\alpha, m), A)} = \sum_{a \in A} p_{(\lambda(\alpha, m), A)}(a) \delta_a.$$

The following result was, with slightly different notation, proved in [2].

**Lemma 4.9.** [2, Proposition 2.15] *For all  $\varepsilon > 0$ , all  $\gamma < \alpha$ , and all  $M \in [\mathbb{N}]^\omega$ , there is an  $N = N(\gamma, \alpha, M, \varepsilon) \in [M]^\omega$ , so that  $\mathbb{P}_{(\alpha, B)}(A) < \varepsilon$  for all  $B \in \text{MAX}(\mathcal{S}_\alpha \cap [N]^\omega)$  and  $A \in \mathcal{S}_\gamma$ .*

If  $\alpha < \omega_1$  and  $A \in \text{MAX}(\mathcal{S}_\alpha)$  we denote the expectation of a function  $f : A \rightarrow \mathbb{R}$  with respect to  $\mathbb{P}_{(\alpha, A)}$  by  $\mathbb{E}_{(\alpha, A)}(f)$ . We finish this section with the following Corollary of Lemma 4.9. It will be used later to estimate the Szlenk index of Banach spaces.

**Corollary 4.10.** *For each  $\alpha < \omega_1$  and  $A \in \text{MAX}(\mathcal{S}_\alpha)$  let  $f_A : A \rightarrow [-1, 1]$  have the property that  $\mathbb{E}_{(\alpha, A)}(f) \geq \rho$ , for some fixed number  $\rho \in [-1, 1]$ . For  $\delta > 0$  and  $M \in [\mathbb{N}]^\omega$  put*

$$\mathcal{A}_{\delta, M} = \left\{ A \in \mathcal{S}_\alpha \cap [M]^{<\omega} : \exists B \in \text{MAX}(\mathcal{S}_\alpha \cap [M]^{<\omega}), A \subset B, \text{ and } f_B(a) \geq \rho - \delta \text{ for all } a \in A \right\}.$$

*Then  $\text{CB}(\mathcal{A}_{\delta, M}) = \omega^\alpha + 1$ .*

*Proof.* Assume our claim is not true. Then we choose  $\gamma < \alpha$  and  $k \in \mathbb{N}$  so that  $\text{CB}(\mathcal{A}_{\delta, M}) < \omega^\gamma k$ . Indeed, if  $\alpha$  is a successor ordinal we choose  $\gamma$  to be the predecessor of  $\alpha$  and  $k \in \mathbb{N}$  large enough and if  $\alpha$  is limit ordinal we choose  $\gamma < \alpha$  large enough and  $k = 1$ . Thus,  $\text{CB}(\mathcal{A}_{\delta, M}) < \text{CB}(\mathcal{S}_{\gamma, k}) = \omega^\gamma k + 1$ . By Lemma 4.7 and the fact that  $\text{CB}(\mathcal{S}_{\gamma, k} \cap [N]^{<\omega}) = \omega^\gamma k + 1$ , for all  $N \in [M]^\omega$ , we deduce that there is an  $N \in [M]^\omega$  so that  $\mathcal{A}_{\delta, N} \subset \mathcal{S}_{\gamma, k}$ .

Let  $0 < \varepsilon < \delta/2k$ . We can use Lemma 4.9 and assume that, after possibly replacing  $N$  by an infinite subset, that  $\mathbb{P}_{(\alpha, B)}(A) < \varepsilon$  for all  $B \in \text{MAX}(\mathcal{S}_\alpha \cap [N]^{<\omega})$  and all  $A \in \mathcal{S}_\gamma \cap [N]^{<\omega}$ . But this implies that for all  $B \in \text{MAX}(\mathcal{S}_\alpha \cap [N]^{<\omega})$  that  $\{b : f_B(b) \geq \rho - \delta\} \in \mathcal{S}_{\gamma, k}$  and thus

$$\mathbb{E}_{(\alpha, B)}(f_B) \leq \rho - \delta + \mathbb{P}_{(\alpha, B)}(\{b \in B : f_B(b) \geq \rho - \delta\}) \leq \rho - \delta + k\varepsilon < \rho - \delta/2$$

which contradicts our assumption on the expected value of  $f_B$ .  $\square$

## 5. THE SZLENK INDEX OF $Z$ AND $W$

Let  $X$  be our space with separable dual and let  $(E_n)$  be a shrinking FMD which together with its biorthogonal sequence  $(F_n)$  satisfies the conclusions of Lemma 2.3. The main goal of this section is to show that the space  $Z$ , as constructed in Section 3, has the same Szlenk index as  $X$ , and that also  $Z^*$  and  $X^*$  share the same Szlenk index if  $X$  is reflexive. Secondly we will prove that the space  $W$  constructed from a space  $V$  with FDD  $(V_j)$  before Theorem 3.9 has the same Szlenk index as  $V$ , and that  $W^*$  and  $V^*$  have the same Szlenk indices if  $V$  is reflexive. We thereby verified part (a) and (b) of our Main Theorem. We first recall the definition and basic properties of the Szlenk index. We then prove further properties that are relevant for our purposes, including the statement of Theorem C.

Let  $K$  be a non-empty bounded subset of  $X^*$ . For  $\varepsilon \geq 0$  the  $\varepsilon$ -derivative of  $K$  is

$$\begin{aligned} K'_\varepsilon &= \left\{ x^* \in X^* : \exists (x_i^*)_{i \in I} \subset K \text{ net, } x_i^* \xrightarrow{w^*}_{i \in I} x^*, \text{ and } \|x^* - x_i\| \geq \varepsilon \right\}, \\ &= \left\{ x^* \in X^* : \exists (x_n^*)_{n \in \mathbb{N}} \subset K \quad x_n^* \xrightarrow{w^*} x^*, \text{ and } \|x^* - x_n\| \geq \varepsilon \right\}. \end{aligned}$$

The second equality follows from the assumption that  $X$  is separable, which yields that the  $w^*$ -topology is metrizable on bounded subsets of  $X^*$ . It is easy to see that  $K'_\varepsilon$  is a  $w^*$ -compact subset of  $\overline{K}^{w^*}$ . Moreover, it is clear that if  $K \subset \tilde{K} \subset X^*$  are bounded, then  $K'_\varepsilon \subset \tilde{K}'_\varepsilon$ , and that  $(rK)'_\varepsilon = r(K'_{\varepsilon/r})$  for  $\varepsilon, r > 0$ . Next, we define for a bounded set  $K \subset X^*$ ,  $\varepsilon > 0$  and an ordinal  $\alpha$  the  $(\alpha, \varepsilon)$ -derivative of  $K$  recursively by

$$K_\varepsilon^{(0)} = K, \quad K_\varepsilon^{(\alpha+1)} = (K_\varepsilon^{(\alpha)})'_\varepsilon \text{ for } \alpha < \omega_1, \text{ and } K_\varepsilon^{(\lambda)} = \bigcap_{\alpha < \lambda} K_\varepsilon^{(\alpha)} \text{ for limit ordinals } \lambda < \omega_1.$$

It was shown in [28] that our assumption that  $X^*$  is separable is equivalent with the property that for every bounded  $K \subset X^*$  the  $\varepsilon$ -Szlenk index of  $K$ , defined by

$$\text{Sz}(K, \varepsilon) < \min\{\alpha < \omega_1 : K_\varepsilon^{(\alpha)} = \emptyset\},$$

exists. We define the Szlenk index of  $K \subset X^*$  and the Szlenk index of  $X$  by

$$\text{Sz}(K) = \sup_{\varepsilon > 0} \text{Sz}(K, \varepsilon) \text{ and } \text{Sz}(X) = \text{Sz}(B_{X^*}) = \sup_{\varepsilon > 0} \text{Sz}(B_{X^*}, \varepsilon).$$

*Remark 5.1.* The original definition of  $K'_\varepsilon$  in [28] is slightly different, and might lead to different  $\varepsilon$ -Szlenk indices. Nevertheless it gives the same values of  $\text{Sz}(K)$  and  $\text{Sz}(X)$ . It is also not hard to see that  $\text{Sz}(X)$ , and more generally  $\text{Sz}(K)$ , for bounded  $K \subset X^*$ , are invariant under renormings of  $X$ .

For our purposes it will also be important to recall the result of [1, Theorems 3.22 and 4.2] which states that  $\text{Sz}(X)$  is always of the form  $\text{Sz}(X) = \omega^\alpha$ , for some  $\alpha < \omega_1$ .

The following equivalent characterization of the Szlenk index is a generalization of [1, Theorem 4.2] where it was proven for the case  $K = B_{X^*}$ . Our proof will be different and uses the properties of the FMD  $(E_n)$ .

**Lemma 5.2.** *For a  $w^*$ -compact set  $K \subset X^*$  and  $0 < c < 1$  we define*

$$\mathcal{F}_c(K) = \mathcal{F}_c(X, (E_j), K) = \left\{ (x_1, x_2, \dots, x_l) \subset S_X : \begin{array}{l} (x_j) \text{ is skipped with resp. to } (E_j), \\ \exists x^* \in K \forall j = 1, 2, \dots, l \quad x^*(x_j) \geq c \end{array} \right\}.$$

*If  $K \neq \emptyset$ , but  $\|x^*\| < c$  for all  $x^* \in K$  and  $x \in S_X$ , put  $\mathcal{F}_c(K) = \{\emptyset\}$  and put  $\mathcal{F}_c(\emptyset) = \emptyset$ . Then*

$$\text{Sz}(K) = \sup_{c > 0} \text{I}_w(\mathcal{F}_c(K)).$$

*Remark 5.3.* In the case that  $K = B_{X^*}$  the set  $\mathcal{F}_c(B_{X^*})$  can be rewritten as

$$\mathcal{F}_c(B_{X^*}) = \left\{ (x_1, x_2, \dots, x_l) \subset S_X : \begin{array}{l} (x_j) \text{ is skipped block with respect to } (E_j) \text{ and} \\ \forall a_1, a_2, \dots, a_l \geq 0 \quad \left\| \sum_{j=1}^l a_j x_j \right\| \geq c \sum_{j=1}^l a_j \end{array} \right\}.$$

Indeed “ $\subset$ ” is clear, while “ $\supset$ ” follows from applying the Hahn Banach Theorem to separate 0 from the convex hull of the set  $\{x_1, x_2, \dots, x_l\}$  for each  $(x_1, x_2, \dots, x_l)$  in the left hand set.

*Proof of Lemma 5.2.* Without loss of generality we assume that  $K \subset B_{X^*}$  and show first for  $0 < \eta < c < 1$  that

$$(29) \quad (\mathcal{F}_c(K))'_w \subset \mathcal{F}_c(K'_{c-\eta}).$$

Let  $(x_1, x_2, \dots, x_l) \in (\mathcal{F}_c(K))'_w$ , and let  $(y_k) \subset S_X$  be a  $w$ -null sequence with  $(x_1, x_2, \dots, x_l, y_k) \in \mathcal{F}_c(K)$ , for  $k \in \mathbb{N}$ . For  $k \in \mathbb{N}$  we choose a  $x_k^* \in K$  such that  $x_k^*(x_i) \geq c$ , for  $i = 1, 2, \dots, l$ ,

and  $x_k^*(y_k) \geq c$ . Without loss of generality we can assume, after passing to subsequences if necessary, that  $x_k^*$  converges in  $w^*$  to some  $x^* \in K$ . We observe that

$$\limsup_{k \rightarrow \infty} \|x_k^* - x^*\| \geq \limsup_{k \rightarrow \infty} (x_k^* - x^*)(y_k) = \limsup_{k \rightarrow \infty} x_k^*(y_k) \geq c,$$

where in the equality we used the assumption that  $(y_k)$  is weakly null. It follows therefore that  $x^* \in K'_{c-\eta}$  and since  $x^*(x_i) = \lim_{k \rightarrow \infty} x_k^*(x_i) \geq c$  for all  $i = 1, 2, \dots, l$  we deduce that  $(x_1, x_2, \dots, x_l) \in \mathcal{F}_c(K'_{c-\eta})$ , which finishes the verification of (29).

Using a straightforward induction argument (29) yields that for all  $\alpha < \omega_1$

$$(\mathcal{F}_c(K))_{\mathbf{w}}^{(\alpha)} \subset \mathcal{F}_c(K_{c-\eta}^{(\alpha)}).$$

In particular this yields that  $K_{c-\eta}^{(\alpha)} \neq \emptyset$  if  $(\mathcal{F}_c(K))_{\mathbf{w}}^{(\alpha)} \neq \emptyset$ . Thus we have  $I_w(\mathcal{F}_c(K)) \leq \text{Sz}(K, c - \eta)$ , for  $0 < \eta < c$  which yields  $\sup_{c > 0} I_w(\mathcal{F}_c(K)) \leq \sup_{c > \eta > 0} \text{Sz}(K, c - \eta) = \text{Sz}(K)$ .

In order to show the reverse inequality we show for  $c < \frac{1}{3}$  and  $\eta < c$  that

$$(30) \quad \mathcal{F}_c(K'_{3c}) \subset (\mathcal{F}_{c-\eta}(K))'_{\mathbf{w}}.$$

Let  $(x_1, x_2, \dots, x_l) \in \mathcal{F}_c(K'_{3c})$  and let  $x^* \in K'_{3c}$  such that  $x^*(x_i) \geq c$ , for  $i = 1, 2, \dots, l$ . We choose a sequence  $(x_k^*) \subset K$  which converges in  $w^*$  to  $x^*$ , and for which  $\|x^* - x_k^*\| \geq 3c$ , for all  $k \in \mathbb{N}$ . Without loss of generality we assume that  $x_k^*(x_i) \geq c - \eta$ , for all  $k \in \mathbb{N}$  and  $i = 1, 2, \dots, l$ .

Since  $(E_n)$  is a shrinking FMD, and thus  $(x^* - x_k^*) \in \overline{\text{span}(F_j : j \in \mathbb{N})}$ , for  $k \in \mathbb{N}$ , we can, after passing to a subsequence, assume that there is a *doubly-skipped* block sequence  $(z_k^*)$  with respect to  $(F_n)$  in  $S_{X^*}$  (meaning  $\max \text{rg}_E(z_k^*) < \min \text{rg}_E(z_{k+1}^*) - 2$ , for  $k \in \mathbb{N}$ ) so that  $\lim_{k \rightarrow \infty} \|z_k^* - (x_k^* - x^*)\| = 0$ . Since  $(E_n)$  and  $(F_n)$  satisfy property (3) of Lemma 2.3 we can, possibly by passing to subsequences, find a block  $(z_k)$  (more precisely:  $\text{rg}_E(z_k) \subset [\min \text{rg}_E(z_k^*) - 1, \max \text{rg}_E(z_k^*) + 1]$ ) with respect to  $(E_n)$  in  $S_X$  so that  $z_k^*(z_k) \geq c \frac{3}{2.5}$ , for all  $k \in \mathbb{N}$ . Since  $(E_n)$  is shrinking  $(z_k)$  is weakly null, and after passing to subsequences again, if necessary, we can assume that

$$x_k^*(z_k) = z_k^*(z_k) + (x_k^* - x^* - z_k^*)(z_k) + x^*(z_k) \geq c - \eta \text{ for all } k \in \mathbb{N}.$$

It follows that  $(x_1, x_2, \dots, x_l, z_k) \in \mathcal{F}_{c-\eta}(K)$  for all large enough  $k \in \mathbb{N}$  and thus it follows that  $(x_1, x_2, \dots, x_l) \in (\mathcal{F}_{c-\eta}(K))'_{\mathbf{w}}$ , since  $(z_k)$  is weakly null, and thus yields (30).

Again by transfinite induction we deduce from (30) that for all  $\alpha < \omega_1$  (recall the notation of  $\mathcal{F}_{\mathbf{w}}^{(\alpha)}$  introduced in Example 4.6 for the derivative with respect to weak null sequences for trees  $\mathcal{F}$  on  $S_X$ )

$$\mathcal{F}_c(K_{3c}^{(\alpha)}) \subset (\mathcal{F}_{c-\eta}(K))_{\mathbf{w}}^{(\alpha)}.$$

This implies in particular that if  $K_{3c}^{(\alpha)}$  is not empty then  $(\mathcal{F}_{c-\eta}(K))_{\mathbf{w}}^{(\alpha)}$  is not empty. Thus  $\text{Sz}(K) = \sup_{c > 0} \text{Sz}(K, c) \leq \sup_{c > 0} I_w(\mathcal{F}_c(K))$ , which finishes our proof.  $\square$

In our next step we prove Theorem C using Corollary 4.10.

*Proof of Theorem C.* Since  $\text{Sz}(X)$  is always of the form  $\omega^\alpha$  for some  $\alpha < \omega_1$ , and since  $\text{Sz}(X) \geq \text{Sz}(K)$ , we have  $\text{Sz}(X) \geq \min\{\omega^\alpha : \omega^\alpha \geq \text{Sz}(K)\}$ .

In order to show the reverse inequality, we first assume without loss of generality that  $K$  is 1-norming  $X$ , because otherwise we could pass to the equivalent norm defined by

$$\|x\| = \sup_{x^* \in K} |x^*(x)| \text{ for } x \in X.$$



Since  $\text{Sz}(K) = \text{Sz}(\overline{K}^{w*})$  we also assume that  $K$  is  $w^*$ -compact. Let  $\alpha < \omega_1$  be such that  $\text{Sz}(X) = \omega^\alpha$ , and assume that our claim is not true and that there is a  $\beta < \alpha$ , so that  $\text{Sz}(K) \leq \omega^\beta$ . By Lemma 5.2  $\omega^\alpha = \text{Sz}(X) = \sup_{c>0} \text{CB}(\mathcal{F}_c(B_{X^*}))$ , where

$$\mathcal{F}_c(B_{X^*}) = \mathcal{F}_c(X, (E_j), B_{X^*}) = \left\{ (x_1, x_2, \dots, x_l) \subset S_X : \begin{array}{l} (x_j) \text{ is skipped with resp. to } (E_j), \\ \exists x^* \in B_{X^*} x^*(x_j) \geq c, j=1, 2 \dots l \end{array} \right\}.$$

Thus, there is  $c \in (0, 1)$  with  $I_w(\mathcal{F}_c(B_{X^*})) > \omega^\beta$ . Since  $\mathcal{F}_c(B_{X^*})$  is hereditary we can apply Proposition 4.4 and Remark 4.5, and choose a family  $(x_F)_{F \in \mathcal{S}_\beta \setminus \{\emptyset\}} \subset S_X$ , so that for every  $F = \{m_1, m_2, \dots, m_l\} \in \mathcal{S}_\beta \setminus \{\emptyset\}$ ,  $\bar{x}_F = (x_{\{m_1\}}, x_{\{m_1, m_2\}}, \dots, x_{\{m_1, m_2, \dots, m_l\}}) \in \mathcal{F}_c(B_{X^*})$ , and so that for every  $F \in \mathcal{S}_\beta \setminus \text{MAX}(\mathcal{S}_\beta)$ ,  $(x_{F \cup \{n\}} : n > \max(F))$  is a weak null sequence (recall that  $\mathcal{S}_\beta = \mathcal{F}_{\omega^\beta}$ ). We now want to apply Corollary 4.10. For every  $B = \{n_1, n_2, \dots, n_l\} \in \text{MAX}(\mathcal{S}_\beta)$ , we have that  $\left\| \sum_{i=1}^l p_{\beta, B}(n_i) x_{\{n_1, n_2, \dots, n_i\}} \right\| \geq c$ . We recall that the probability  $\mathbb{P}_{\beta, B}$  on  $B$ , with coefficients  $p_{\beta, B}(n_i)$ ,  $i = 1, 2, \dots, l$ , where introduced in Section 4 before Lemma 4.8. Since  $K$  is 1-norming and compact we choose to every  $B = \{n_1, n_2, \dots, n_l\} \in \text{MAX}(\mathcal{S}_\beta)$  an element  $x_B^* \in K$ , so that

$$x_B^* \left( \sum_{i=1}^l p_{\beta, B}(n_i) x_{\{n_1, n_2, \dots, n_i\}} \right) = \left\| \sum_{i=1}^l p_{\beta, B}(n_i) x_{\{n_1, n_2, \dots, n_i\}} \right\| \geq c \text{ for all } i = 1, 2, \dots, l.$$

For every  $B = \{n_1, n_2, \dots, n_l\} \in \text{MAX}(\mathcal{S}_\beta)$  we define  $f_B : B \rightarrow [-1, 1]$ ,  $n_i \mapsto x_B^*(x_{\{n_1, n_2, \dots, n_i\}})$ , and note that we can apply Corollary 4.10 to the family  $(f_B : B \in \text{MAX}(\mathcal{S}_\beta))$ , and obtain an  $M \in [\mathbb{N}]^\omega$  so that for  $\delta = c/2$  we have  $\text{CB}(\mathcal{A}_{\delta, M}) = \omega^\beta + 1$ , where

$$\mathcal{A}_{\delta, M} = \left\{ A \in \mathcal{S}_\beta \cap [M]^{<\omega} : \exists B \in \text{MAX}(\mathcal{S}_\beta \cap [M]^{<\omega}), A \subset B, \text{ and } f_B(a) \geq \rho - \delta \text{ for all } a \in A \right\}.$$

We now verify (ii) of Proposition 4.4 for the hereditary tree  $\mathcal{F}_{c/2}(K)$ , in order to conclude that  $I_w(\mathcal{F}_{c/2}(K)) > \omega^\beta$ , which would be a contradiction to the assumption that  $\text{Sz}(K) = \sup_{r>0} I_w(\mathcal{F}_r(K)) \leq \omega^\beta$  (for the equality see Lemma 5.2).

By Remark 4.5 we find an order isomorphism  $\pi : \mathcal{S}_\beta = \mathcal{F}_{\omega^\beta} \rightarrow \mathcal{A}_{\delta, M}$ , so that  $\pi(A \cup \{n\}) \setminus \pi(A) = \{\max \pi(A \cup \{n\})\}$  for all  $A \in \mathcal{S}_\beta \setminus \text{MAX}(\mathcal{S}_\beta)$ , and all  $n < \max(A)$ .

Then define for  $F = \{m_1, m_2, \dots, m_l\} \in \mathcal{S}_\beta \setminus \{\emptyset\}$ ,  $z_F = x_{\pi(F)} \in S_X$ . Since  $\pi(F) \in \mathcal{A}_{\delta, M}$ , there is a maximal  $B$  in  $\mathcal{S}_\beta \cap [M]^{<\omega}$  containing  $\pi(B)$ , so that  $x_B^*(z_{\{m_1, m_2, \dots, m_i\}}) \geq c/2$  for all  $i = 1, 2, \dots, l$ . It follows therefore that  $\bar{z}_F = (z_{\{m_1\}}, z_{\{m_1, m_2\}}, \dots, z_{\{m_1, m_2, \dots, m_l\}}) \in \mathcal{F}_{c/2}(K)$  for all  $F \in \mathcal{S}_\beta \setminus \{\emptyset\}$ . Secondly, it follows for any non maximal  $F \in \mathcal{S}_\beta$ , that  $(z_{F \cup \{n\}} : n > \max(F)) = (x_{\pi(F \cup \{n\})} : n > \max(F)) = (x_{\pi(F) \cup \{\max(\pi(F \cup \{n\}))\}} : n > \max(F))$  is weakly null. This verifies that  $(\bar{z}_F : F \setminus \{\emptyset\})$  satisfies the conditions in (ii) of Proposition 4.4 and finishes the proof.  $\square$

*Remark 5.4.* Theorem C states that if  $\text{Sz}(X) = \omega^\alpha$  then for any  $\beta < \alpha$  and set  $K \subset B_{X^*}$ , which norms  $X$ , it follows that  $\text{Sz}(K) > \omega^\beta$ . This is the optimal estimate we have for the Szlenk index of a norming set  $K$ . Indeed, if  $X = C[0, \omega^{\omega^\alpha}]$  then it follows by [27, Théorème, p.91] that  $\text{Sz}(X) = \omega^{\alpha+1}$ . The set  $K = \{\delta_\gamma : \gamma \in [0, \omega^{\omega^\alpha}]\}$  of Dirac measures on  $[0, \omega^{\omega^\alpha}]$  is norming  $X$ , and  $\text{Sz}(K)$  equals to the Cantor Bendixson index of  $[0, \omega^{\omega^\alpha}]$  which is  $\omega^{\alpha+1}$ .

We are now in the position to compute the Szlenk index of the space  $Z$ , which was constructed in Section 3. We recall the definition of the sets  $D^* \subset X^*$ ,  $B \subset X^*$ ,  $\mathbb{D}^* \subset Z^*$ ,  $\mathbb{B}^* \subset B_{Z^*}$ , the spaces  $U_{\vec{j}}$ ,  $\vec{j} = (j_k) \in \prod_{k=1}^\infty [n_k, n_{k+1}]$ , and the embedding  $I : X \rightarrow Z$  defined in and before Proposition 3.3.

**Lemma 5.5.** *For  $K \subset \mathbb{B}^*$ , bounded, and  $\eta > 0$*

$$I^*(K'_c) \subset (I^*(K))'_{c-\eta}.$$

*For  $\alpha < \omega_1$  it follows that*

$$I^*(K_c^{(\alpha)}) \subset (I^*(K))^{(\alpha)}_{c-\eta}.$$

*Proof.* Assume  $z^* \in K'_c$  and  $\eta > 0$ . Let  $(z^*(n) : n \in \mathbb{N}) \subset K$ , with  $z^*(n) \rightarrow_{n \rightarrow \infty} z^*$  with respect to the  $w^*$ -topology in  $Z^*$ , and  $\|z^* - z^*(n)\| \geq c$  for all  $n \in \mathbb{N}$ . Write  $z^*$  and  $z^*(n)$  as  $z^* = (x_k^* : k \in \mathbb{N}) \in \mathbb{B}^*$  and  $z^*(n) = (x_k^*(n) : k \in \mathbb{N}) \in \mathbb{B}^*$ , and let  $x^* = I^*(z^*) = w^* - \lim_{n \rightarrow \infty} I^*(z^*(n))$ , and  $x^*(n) = I^*(z^*(n))$ , for all  $n \in \mathbb{N}$ . After passing to subsequences we can assume that there is a sequence  $\bar{j} = (j_k) \subset \mathbb{N}$ , with  $j_k \in [n_k, n_{k+1}]$  so that  $\text{rg}_E(x_k^*(n)) \subset (j_{k-1}, j_k)$ , for all  $k \in \mathbb{N}$  and all  $n \geq k$ , and thus  $\text{rg}_E(x^*) \subset (j_{k-1}, j_k)$ , for all  $k \in \mathbb{N}$ . Since  $x^*$  and  $x^*(n)$  are in  $U_{\bar{j}}$  it follows from Proposition 3.3 part (4) that  $\|x^*(n) - x^*\| = \|z^*(n) - z^*\| \geq c$ . Since  $x^* = \lim_{n \rightarrow \infty} x^*(n)$  and  $x^*(n) \in I^*(K)$ , for  $n \in \mathbb{N}$ , it follows that  $x^* \in (I^*(K))'_c$ , which of proof the first claim. The second claim follows by transfinite induction for all  $\alpha < \omega_1$ .  $\square$

**Corollary 5.6.**  $\text{Sz}(X) = \text{Sz}(Z)$ .

*Proof.* We apply the second statement of Lemma 5.5 to  $K = \mathbb{B}^*$  and deduce from it that  $\text{Sz}(\mathbb{B}^*) \leq \text{Sz}(B^*)$ , since by Proposition 3.3  $I^*(\mathbb{B}^*) = B^*$ . If  $\alpha$  is such that  $\text{Sz}(X) = \omega^\alpha$ , it follows from the fact that  $\mathbb{B}^*$  is norming  $Z$  and Theorem C that  $\text{Sz}(Z) \leq \omega^\alpha$ , and thus, since  $X \subset Z$ , that  $\text{Sz}(Z) = \text{Sz}(X)$ .  $\square$

**Lemma 5.7.** *If  $X$  is reflexive then  $\text{Sz}(X^*) = \text{Sz}(Z^*)$ .*

*Proof.* Since  $X$  and  $Z$  are reflexive we can change the roles of  $X$  and  $X^*$  and of  $Z$  and  $Z^*$  in Lemma 5.2 and deduce that

$$\text{Sz}(X^*) = \sup_{c>0} \text{I}_w(\mathcal{F}_c(X^*, (F_j), B_X)),$$

where

$$\mathcal{F}_c(X^*, (F_j), B_X) = \left\{ (x_1^*, x_2^*, \dots, x_l^*) \subset S_{X^*} : \begin{array}{l} (x_j^*) \text{ is skipped with resp. to } (F_j), \\ \exists x \in B_X \forall i = 1, 2, \dots, l \quad x_i^*(x) \geq c \end{array} \right\}$$

and

$$\text{Sz}(Z^*) = \sup_{c>0} \text{I}_w(\mathcal{F}_c(Z^*, (Z_j^*), B_Z)),$$

where

$$\mathcal{F}_c(Z^*, (Z_j^*), B_Z) = \left\{ (z_1^*, z_2^*, \dots, z_l^*) \subset S_{Z^*} : \begin{array}{l} (z_j^*) \text{ is skipped with resp. to } (Z_j^*), \\ \exists z \in B_Z \forall i = 1, 2, \dots, l \quad z_i^*(z) \geq c \end{array} \right\}.$$

We will abbreviate  $\mathcal{F}_c^{Z^*} = \mathcal{F}_c(Z^*, (Z_j^*), B_Z)$  and  $\mathcal{F}_c^{X^*} = \mathcal{F}_c(X^*, (F_j), B_X)$  and show that for  $0 < c < 1/3$  and for  $0 < \eta < c/3$

$$(31) \quad \text{I}_w(\mathcal{F}_c^{Z^*}) \leq \text{I}_w(\mathcal{F}_{c/3-\eta}^{X^*})$$

which, using Lemma 5.2, yields the statement of our lemma. We first prove the following

**Claim 1.** If  $(z_j^*)_{j=1}^l \in \mathcal{F}_c^{Z^*}$ , then there exists a sequence  $(y_j^*)_{j=1}^l \subset \mathbb{D}^*$  so that  $\text{rg}_Z(y_j^*) \subset \text{rg}_Z(z_j^*)$ , for  $j = 1, 2, \dots, l$  (and thus  $(y_j^*)_{j=1}^l$  is also a skipped sequence with respect to  $Z_i^*$ ) and so that the sequence  $(I^*(y_j^*))_{j=1}^l$  is in  $\mathcal{F}_{c/3}^{X^*}$ .

To show Claim 1, let  $(z_j^*)_{j=1}^l \in \mathcal{F}_c^{Z^*}$  and let  $z \in B_Z$  so that  $z_j^*(z) \geq c$ , for all  $j = 1, 2, \dots, l$ . For  $j = 1, 2, \dots, l$  let  $z_j = P_{\text{rg}_Z(z_j^*)}^Z$ . Since  $\mathbb{B}^*$  is 1-norming  $Z$  there are  $\tilde{y}_j^* \in \mathbb{B}^*$ ,  $j = 1, 2, \dots, l$ , so that  $\tilde{y}_j^*(z_j) = \|z_j\| \geq z_j^*(z) \geq c$ . We define  $y_j^* = P_{\text{rg}_Z(z_j^*)}^{(Z^*)}(\tilde{y}_j^*) / \|P_{\text{rg}_Z(z_j^*)}^{(Z^*)}(\tilde{y}_j^*)\|$ , for  $j = 1, 2, \dots, l$ . Then  $y_j^* \in \mathbb{B}^*$ , and since the projection constant of  $(Z_j)$  does not exceed the value 3, it follows for all  $j = 1, 2, \dots, l$ , that

$$y_j^*(z) = y_j^*(P_{\text{rg}_Z(z_j^*)}^{Z^*}(z)) = y_j^*(z_j) \geq \frac{c}{3}.$$

Since  $(y_j^*)_{j=1}^l$  is a skipped block sequence with respect to  $(Z_j^*)$ , which is in  $\mathbb{D}^*$ , Proposition 3.3 (4) yields that the sequence  $(x_j^*)_{j=1}^l = (I^*(y_j^*))_{j=1}^l$  is a skipped block sequence in  $D^*$  with respect to  $(F_j)$  which is isometrically equivalent to  $(y_j^*)_{j=1}^l$ . It follows therefore that for all  $(a_j)_{j=1}^l \subset [0, \infty)$  we have

$$\left\| \sum_{j=1}^l a_j x_j^* \right\| = \left\| \sum_{j=1}^l a_j y_j^* \right\| \geq \sum_{j=1}^l a_j y_j^*(z) \geq \frac{c}{3} \sum_{j=1}^l a_j.$$

By the Remark 5.3 this yields that  $(x_j^*)_{j=1}^l \in \mathcal{F}_c$  and thus proves our claim.

**Claim 2.** We will prove by transfinite induction for all  $\alpha \geq 0$  that if  $(z_j^*)_{j=1}^l$  is a skipped normalized block sequence with respect to  $(Z_j^*)$  and  $I_w(\mathcal{F}_c^{Z^*}(z_1^*, z_2^*, \dots, z_l^*)) \geq \alpha + 1$ , then there is a sequence  $(y_j^*)_{j=1}^l \in \mathbb{D}^*$ , with  $\text{rg}_Z(y_j^*) \subset \text{rg}_Z(z_j^*)$ , and so that for  $(x_j^*)_{j=1}^l = (I^*(y_j^*))_{j=1}^l$  and all  $0 < \eta < c/3$  it follows that  $I_w(\mathcal{F}_{c/3-\eta}^{X^*}(x_1^*, x_2^*, \dots, x_l^*)) \geq \alpha + 1$ .

For  $\alpha = 0$  Claim 2 reduces to Claim 1 since by (22)  $I_w(\mathcal{F}_c^{Z^*}(z_1^*, z_2^*, \dots, z_l^*)) \geq 1$  means that  $(z_j^*)_{j=1}^l \in \mathcal{F}_c^{Z^*}$ . Assume that Claim 2 is true for  $\alpha$  and let  $0 < \eta < c/3$  and  $(z_j^*)_{j=1}^l$  be a skipped normalized block sequence with respect to  $(Z_j^*)$  for which  $I_w(\mathcal{F}_c^{Z^*}(z_1^*, z_2^*, \dots, z_l^*)) \geq \alpha + 2$ . It follows from (23) that there is a weakly null sequence  $(z_{l+1}^*(n))_{n \in \mathbb{N}} \subset S_{Z^*}$  so that  $I_w(\mathcal{F}_c^{Z^*}(z_1^*, z_2^*, \dots, z_l^*, z_{l+1}^*(n))) \geq \alpha + 1$  for all  $n \in \mathbb{N}$ . By our induction hypothesis (for  $\eta/3$  instead of  $\eta$ ) we can find for  $n \in \mathbb{N}$  a sequence  $(y_1^*(n), y_2^*(n), \dots, y_l^*(n), y_{l+1}^*(n))$  in  $\mathbb{B}^*$ , so that  $\text{rg}_Z(y_j^*(n)) \subset \text{rg}_Z(z_j^*)$ , for  $j = 1, 2, \dots, l$ , and  $\text{rg}_Z(y_{l+1}^*(n)) \subset \text{rg}_Z(z_{l+1}^*(n))$ , and so that for  $(x_j^*(n))_{j=1}^{l+1} = (I^*(y_j^*(n)))_{j=1}^{l+1}$  it follows that  $I_w(\mathcal{F}_{(c-\eta)/3}^X(x_1^*(n), x_2^*(n), \dots, x_{l+1}^*(n))) \geq \alpha + 1$ , for all  $\eta \in (0, c/3)$ .

After passing to subsequences we can assume that  $x_j^* = \lim_{n \in \mathbb{N}} x_j^*(n)$  exists (in norm, because the ranges are bounded) for  $j = 1, \dots, l$ . Using this convergence, we can choose  $n_0 \in \mathbb{N}$ , large enough, so that for all  $n \geq n_0$ , for all sequences  $(x_{l+2}^*, x_{l+3}^*, \dots, x_L^*) \in \mathcal{F}_{(c-\eta)/3}^X(x_1^*(n), x_2^*(n), \dots, x_{l+1}^*(n))$ , and for all numbers  $a_j \geq 0$ ,  $j = 1, 2, \dots, L$ , and we have

$$\begin{aligned} & \left\| \sum_{j=1}^l a_j x_j^* + a_{l+1} x_{l+1}^*(n) + \sum_{j=l+2}^L a_j x_j^* \right\| \\ & \geq \left\| \sum_{j=1}^l a_j x_j^*(n) + a_{l+1} x_{l+1}^*(n) + \sum_{j=l+2}^L a_j x_j^*(n) \right\| - \frac{\eta}{2} \sum_{j=1}^{L+2} a_i \\ & \geq \left( \frac{c-\eta}{3} - \frac{\eta}{2} \right) \sum_{j=1}^{L+2} a_i \geq \left( \frac{c}{3} - \eta \right) \sum_{j=1}^{L+2} a_i, \end{aligned}$$

which proves that

$$\mathcal{F}_{(c-\eta)/3}^X(x_1^*(n), x_2^*(n), \dots, x_{l+1}^*(n)) \subset \mathcal{F}_{(c/3)-\eta}^X(x_1^*, x_2^*, \dots, x_l^*, x_{l+1}^*(n)),$$

and thus

$$I_w(\mathcal{F}_{(c/3)-\eta}^X(x_1^*, x_2^*, \dots, x_l^*, x_{l+1}^*(n))) \geq I_w(\mathcal{F}_{(c-\eta)/3}^X(x_1^*(n), x_2^*(n), \dots, x_{l+1}^*(n))) \geq \alpha + 1,$$

and therefore  $x_{l+1}^*(n) \in (\mathcal{F}_{(c/3)-\eta}^X(x_1^*, x_2^*, \dots, x_l^*))^{(\alpha)}$ , for  $n \in \mathbb{N}$ . Since  $(x_{l+1}^*(n))_{n=1}^\infty$  is weakly null (being a bounded block sequence in a reflexive space), it follows that  $\emptyset \in (\mathcal{F}_{(c/3)-\eta}^X(x_1^*, x_2^*, \dots, x_l^*))^{(\alpha+1)}$ , and thus  $I_w(\mathcal{F}_{(c/3)-\eta}^X(x_1^*, x_2^*, \dots, x_l^*)) \geq \alpha + 2$ , which finishes the induction step in the case of successor ordinals.

If  $\alpha$  is a limit ordinal and  $0 < \eta < c/3$  and if  $(z_j^*)_{j=1}^l$  is a skipped normalized block sequence with respect to  $(Z_j^*)$  with  $I_w(\mathcal{F}_c^{Z^*}(z_1^*, z_2^*, \dots, z_l^*)) \geq \alpha + 1$  we proceed as follows. For every  $\beta < \alpha$  we find by our induction hypothesis a sequence  $(y_j^*(\beta))_{j=1}^l \in \mathbb{D}^*$ , which satisfies the conclusion of Claim 1 and so that for  $(x_j^*(\beta))_{j=1}^l = (I^*(y_j^*(\beta)))_{j=1}^l$  it follows that  $I_w(\mathcal{F}_{(c-\eta)/3}^{X^*}(x_1^*(\beta), x_2^*(\beta), \dots, x_l^*(\beta))) \geq \beta$ . We can assume that  $x_j^* = \lim_{n \rightarrow \infty} x_j^*(\beta_n)$  exists for all  $j = 1, 2, \dots, l$  and for some sequence  $(\beta_n) \subset (0, \alpha)$  which increases to  $\alpha$ . A similar argument as in the successor case shows that we can assume after passing to subsequences that for all  $n \in \mathbb{N}$

$$\mathcal{F}_{(c-\eta)/3}^X(x_1^*(\beta_n), x_2^*(\beta_n), \dots, x_l^*(\beta_n)) \subset \mathcal{F}_{(c/3)-\eta}^X(x_1^*, x_2^*, \dots, x_l^*),$$

and thus that

$$\emptyset \in \bigcap_{n \in \mathbb{N}} (\mathcal{F}_{(c/3)-\eta}^X(x_1^*, x_2^*, \dots, x_l^*))^{(\beta_n)} = (\mathcal{F}_{(c/3)-\eta}^X(x_1^*, x_2^*, \dots, x_l^*))^{(\alpha)},$$

which yields our claim, and finishes the induction claim.

The inequality (31) follows from Claim 2 applied to the empty sequence.  $\square$

The next result proves part (a) and (b) of Theorem B, and applied to the space  $V = Z$  finishes the proof of (a) and (b) of the Main Theorem.

**Lemma 5.8.** *Let  $V$  be a Banach space with shrinking FDD  $(V_j)$ , and let  $W$  be the space with shrinking basis containing  $V$ , which was constructed before Theorem 3.9.*

*Then  $\text{Sz}(W) = \text{Sz}(V)$  and, if  $V$  is reflexive, then  $\text{Sz}(W^*) = \text{Sz}(V^*)$ .*

*Proof.* Let  $\Gamma$  be the set defined before Theorem 3.9 and  $(e_\gamma : \Gamma)$  the basis of  $W$  as introduced there. For  $n \in \mathbb{N}$  let  $W_n = \text{span}(e_{(n,j)} : j \leq l_n)$ , for  $n \in \mathbb{N}$ . As noted in the proof of Theorem 3.9,  $(W_n)$  is an FDD of  $W$ . Like in the proof of Theorem 3.9 we can assume that the set

$$A = \left\{ \sum a_n x_{(n,i_n)}^* : (i_n) \in \prod_{n=1}^\infty \{1, 2, \dots, l_n\}, \left\| \sum_{n=1}^\infty a_n x_{(n,i_n)}^* \right\| \leq 1 \right\}$$

is 1 norming the space  $V$ . The set  $B = \{ \sum_{n=1}^\infty a_n e_{(n,i_n)}^* : \sum_{n=1}^\infty a_n x_{(n,i_n)}^* \in A \}$  is (by definition) 1-norming  $W$ . We also recall the fact, which was obtained in the proof of Theorem 3.9, that if  $(w_j)$  is in  $B$  and is a block sequence with respect to  $(W_n)$ , then the sequence  $(w_j|_V)$  is in  $A$  and it is a block sequence with respect to  $(V_n)$  which is isometrically equivalent to  $(w_j)$ .

The proof that  $\text{Sz}(W) = \text{Sz}(V)$  is very similar to the proof that  $\text{Sz}(Z) = \text{Sz}(X)$ , only a little bit easier since  $W$  and  $V$  have an FDD. We therefore will only sketch it. Let  $\alpha < \omega_1$

so that  $Sz(V) = \omega^\alpha$ . By Theorem C it is enough to show that  $Sz(B) \leq \omega^\alpha$ . In order to accomplish that we first show that for any compact  $K \subset B$ , any  $0 < \eta < c < 1$

$$(32) \quad J^*(K'_c) \subset (J^*(K))'_{c-\eta}$$

where  $J : V \rightarrow W$  is the embedding, and thus  $J^* : W^* \rightarrow V^*$  is the restriction operator. Using the fact that  $J^*$  is  $w^*$ -continuous and the fact that  $R$  maps block sequences in  $B$  into isometrically equivalent block sequences in  $A$  (which was shown within the proof of Theorem 3.9), this can be done the same way we proved Lemma 5.5. From (32) we then deduce by transfinite induction for all  $\alpha < \omega_1$ , and  $0 < \eta < c$  that  $R(B_c^{(\alpha)}) \subset (J^*(B))_{c-\eta}^{(\alpha)} \subset A_{c-\eta}^{(\alpha)}$ , and thus that  $Sz(B) \leq Sz(A) \leq \omega^\alpha$ .

Now assume that  $V$  is reflexive. The verification that  $Sz(W^*) = Sz(V^*)$  is similar to the proof of Lemma 5.7, and again easier since we are dealing now with FDDs. We will therefore also only sketch it. As in Lemma 5.7 we define for  $0 < c < 1$

$$\mathcal{F}_c^{V^*} = \mathcal{F}_c(V^*, (V_j^*), B_V) = \left\{ (v_1^*, v_2^*, \dots, v_l^*) \subset S_{V^*} : \begin{array}{l} (z_j^*) \text{ is skipped with resp. to } (V_j^*), \\ \exists v \in B_V \forall i = 1, 2, \dots, l \quad v_i^*(v) \geq c \end{array} \right\},$$

and

$$\mathcal{F}_c^{W^*} = \mathcal{F}_c(W^*, (W_j^*), B_W) = \left\{ (w_1^*, w_2^*, \dots, w_l^*) \subset S_{W^*} : \begin{array}{l} (w_j^*) \text{ is skipped with resp. to } (W_j^*), \\ \exists w \in B_W \forall i = 1, 2, \dots, l \quad w_i^*(w) \geq c \end{array} \right\},$$

and have to show that for any  $0 < \eta < c$

$$(33) \quad I_w(\mathcal{F}_c^{W^*}) \leq I_w(\mathcal{F}_c^{V^*}).$$

Lemma 5.2 will then yield that  $Sz(V^*) = Sz(W^*)$ .

First we prove, as in Lemma 5.7, the following claim:

**Claim 1:** If  $(w_j^*)_{j=1}^l \in \mathcal{F}_c^{W^*}$ , then there is a sequence  $(\tilde{w}_j^*)_{j=1}^l$  in  $B$  so that  $\text{rg}_W(\tilde{w}_j^*) \subset \text{rg}_W(w_j^*)$ , for  $j = 1, 2, \dots, l$ , and so that  $(J^*(\tilde{w}_j^*))_{j=1}^l \in \mathcal{F}_c^{V^*}$ .

To show Claim 1 we will use the bimonotonicity of  $(V_j)$  and  $(W_j)$  in  $V$  and  $W$ , respectively, the fact that  $A$  and  $B$  are 1 norming  $V$  and  $W$ , respectively, and the fact that  $J^*$  is  $w^*$ -continuous, and maps block sequences of  $B$  to isometrically equivalent block sequences in  $A$ .

Secondly we show, analogously to the proof of Lemma 5.7, by transfinite induction for all  $\alpha < \omega_1$ , that if  $(w_j^*)_{j=1}^l \in \mathcal{F}_c^{W^*}$  is a skipped block basis, for which  $I_w(\mathcal{F}_c^{W^*}(w_1^*, \dots, w_l^*)) \geq \alpha + 1$ , then there is a sequence  $(\tilde{w}_j^*)_{j=1}^l$  in  $B$  for which  $\text{rg}_W(\tilde{w}_j^*) \subset \text{rg}_W(w_j^*)$ , for  $j = 1, 2, \dots, l$ , and for which  $I_w(\mathcal{F}_{c-\eta}^{V^*}(J^*(\tilde{w}_1^*), \dots, J^*(\tilde{w}_l^*))) \geq \alpha + 1$ . (33) follows then by applying Claim 2 to the empty sequence.  $\square$

*Remark 5.9.* Since in Lemma 5.8  $(V_j)$  and  $(W_j)$  are FDDs (and not only FMDs) it was actually unnecessary to define the elements of  $\mathcal{F}_c^{V^*}$  and  $\mathcal{F}_c^{W^*}$  to be skipped block bases, in order to obtain the second part of Lemma 5.8. Nevertheless in order for the argument to also hold in general FMDs, it is necessary to use skipped block bases.

## 6. INFINITE ASYMPTOTIC GAMES WITH RESPECT TO FMDs

In this section we present *Infinite Asymptotic Games*, and show how to use them to deduce embedding results. They were introduced for spaces with FDD in [22, 23]. The name *Infinite Asymptotic Games* was coined by Rosendal [26] who generalized them to a more general setting. In this section we present another generalization of *Infinite Asymptotic Games* by defining them with respect to Finite Dimensional Markushevich Decompositions,

and deduce as in the FDD case a combinatorial principle (see Theorem 6.11), which can be used to characterize the property that a certain Banach embeds into a space with a certain FDD or basis. One of these results is the intrinsic characterization of subspaces of spaces with an unconditional basis by Johnson and Zhang [16, 17]. We will show that if our space  $X$  has the *Unconditional Tree Property*, as defined in [16, 17], and if we have started out with an appropriately blocked FMD, then the FDD  $(Z_i)$  of the space  $Z$  constructed in Section 3 is automatically unconditional. This will lead to an alternate proof of Johnson's and Zhang's results. Actually, some of the ideas, for example the idea of using FMDs instead of FDDs can already be found in their second paper [17]. Nevertheless, since we suspect that Theorem 6.11 could lead to other interesting embedding results, we would like to present it in a more general form. Some of our arguments will be very similar to the arguments in [22, 23]. But for the sake of a better readability and for being self-contained we present the complete arguments.

We start with a general separable Banach space  $X$  and we assume that we have chosen a fixed but, for the moment arbitrary, 1-norming FMD  $(E_n)$  and denote its biorthogonal sequence by  $(F_j)$ . We denote by  $\mathcal{B}_\omega = \mathcal{B}_\omega(X, E)$ ,  $\mathcal{B}_f(X, E)$ , and  $\mathcal{B}_n(X, E)$ ,  $n \in \mathbb{N} \cup \{0\}$ , the set of infinite, or finite sequences, or sequences of length  $n$  in  $S_X \cap c_{00}(E_j)$  which are block sequences with respect to  $(E_j)$  (we require now that also the last element of a sequences  $(x_j)_{j=1}^l \in \mathcal{B}_l$  has finite support).

We consider on  $\mathcal{B}_\omega$  the product topology of the norm topology on  $\mathcal{B}_1 \equiv S_X \cap c_{00}(E_j)$  and denote the closure for  $\mathcal{A} \subset \mathcal{B}_\omega$  with respect to that topology by  $\overline{\mathcal{A}}$ . Note that  $\mathcal{A} \subset \mathcal{B}_\omega$  is open if and only if for  $(x_j) \in \mathcal{B}_\omega$

$$(34) \quad (x_j : j \in \mathbb{N}) \in \mathcal{A} \iff \exists n \in \mathbb{N}, \delta > 0 \quad \{(z_i) \in \mathcal{B}_\omega : \|x_i - z_i\| \leq \delta, i = 1, 2, \dots, n\} \subset \mathcal{A}$$

and  $\mathcal{A}$  is closed if and only if for every  $(x_j) \in \mathcal{B}_\omega$

$$(35) \quad (x_j : j \in \mathbb{N}) \in \mathcal{A} \iff \forall n \in \mathbb{N}, \delta > 0 \exists (z_i) \in \mathcal{A} \quad \|z_i - x_i\| \leq \delta, \text{ for all } i = 1, 2, \dots, n.$$

For  $\mathcal{A} \subset \mathcal{B}_\omega$  and a sequence  $\overline{\varepsilon} = (\varepsilon_j) \subset \mathbb{R}^+$  we define the  $\overline{\varepsilon}$ -fattening of  $\mathcal{A}$  by

$$\mathcal{A}_{\overline{\varepsilon}} = \{(x_j) \in \mathcal{B}_\omega : \exists (z_j) \in \mathcal{A} \quad \|x_j - z_j\| \leq \varepsilon_j\}.$$

For  $\mathcal{A} \subset \mathcal{B}_\omega$  and  $\overline{x} = (x_1, \dots, x_n) \in \mathcal{B}_f$  we define

$$\mathcal{A}(\overline{x}) = \{\overline{z} \in \mathcal{A} : \overline{z} \succ \overline{x}\}.$$

Here we mean, as in Section 2, by  $\overline{z} \prec \overline{x}$  that  $\overline{z}$  is an extension of  $\overline{x}$ .

For  $\mathcal{A} \subset \mathcal{B}_\omega$  we now consider the following  $\mathcal{A}$ -game between two players:

Player I chooses  $k_1 \in \mathbb{N}$ , then Player II chooses  $x_1 \in S_X \cap c_{00}(E_j)$  with  $\min \text{supp}_E(x_1) \geq k_1$ , then again Player I chooses  $k_2 \in \mathbb{N}$ ,  $k_2 > \max \text{rg}_E(x_1)$  and Player II chooses  $x_2 \in S_X \cap c_{00}(E_j)$  with  $\min \text{supp}_E(x_2) \geq k_2$ . This goes on for infinitely many steps and Player I is declared the winner of that game if the resulting sequence  $(x_j)$  lies in  $\mathcal{A}$ .

Let us precisely formulate what it means that Player II has a winning strategy for that  $\mathcal{A}$ -game and observe what a winning strategy is. In order to do so, we define the *full tree on  $\mathbb{N}$  by  $\mathcal{T} = [\mathbb{N}]^{<\omega} = \{A \subset \mathbb{N}, \text{ finite}\}$* . On  $\mathcal{T}$  we consider the order of extensions  $\succ$  introduced in Section 4. A *full indexed tree* will be a family indexed by  $\mathcal{T}$ , in our cases with values in a Banach space  $X$ . For simplicity we will in this section often call a full indexed tree  $(x_t : t \in \mathcal{T}) \subset X$  simply a *tree in  $X$*  if it can not be confused with the type of trees which were considered in Section 4. If  $(x_t)_{t \in \mathcal{T}}$  is a full indexed tree in  $X$  and  $t \in \mathcal{T}$ , we call the sequence  $(x_{(t,k)})_{k > \max(t)}$  a *node of  $(x_t)_{t \in \mathcal{T}}$*  and if  $k_1 < k_2 < k_3 < \dots$  we call the sequence

$(x_{\{k_1, k_2, \dots, k_j\}} : j \in \mathbb{N})$  a branch of  $(x_t)_{t \in \mathcal{T}}$  (note that  $x_\emptyset$  is not part of a branch). The tree  $(x_t)_{t \in \mathcal{T}}$  is called *normalized* if  $(x_t)_{t \in \mathcal{T}} \subset S_X$ , *weakly null* if every node is weakly null and a *block tree with respect to the FMD  $(E_j)$* , if every node is a block sequence with respect to  $(E_j)$ . More generally, if  $\mathcal{U}$  is any topology on  $X$  (for example  $\sigma(X, Y)$  for some  $Y \subset X^*$ ), a  $\mathcal{U}$ -null tree is a tree for which all nodes are  $\mathcal{U}$ -null.

An *indexed subtree* of a tree  $(x_t : t \in \mathcal{T})$  is a family  $(x_t : t \in \mathcal{S})$  indexed by a non empty subset  $\mathcal{S}$  of  $\mathcal{T}$ , which is a subtree of  $\mathcal{T}$ , i.e., which is closed under taking restrictions. We call such an indexed subtree *well-founded* if  $\mathcal{S}$  is well founded (see Section 3). We say that  $(x_t : t \in \mathcal{S})$  is *infinitely branching* if every non maximal  $s \in \mathcal{S}$  has infinitely many direct successors. Assume that  $(x_t : t \in \mathcal{T})$  is a full indexed tree and that  $\mathcal{T}' \subset \mathcal{T}$ , is a subtree which has the property that for all  $t \in \mathcal{T}'$  the set  $\{n \in \mathbb{N} : \{t, n\} \in \mathcal{T}'\}$  is infinite. We call then  $(x_t : t \in \mathcal{T}')$  a *full indexed subtree* of  $(x_t : t \in \mathcal{T})$ . It is easy to see that, that there is an order isomorphism between  $\mathcal{T}'$  and  $\mathcal{T}$ , and, using that order isomorphism, we can reorder  $(x_t : t \in \mathcal{T}')$  into  $(z_t : t \in \mathcal{T})$ , having the same branches and nodes as  $(x_t : t \in \mathcal{T}')$ . In that case we also call  $(z_t : t \in \mathcal{T})$  a full indexed subtree of  $(x_t : t \in \mathcal{T})$ .

**Proposition 6.1.** *Assume that  $Y$  is a subspace of  $X^*$  which separates points of  $X$ . For example  $Y$  could be the closed linear span of the biorthogonal sequence  $(F_j)$ .*

*Let  $(x_t : t \in \mathcal{T}) \subset S_X$  be a normalized  $\sigma(X, Y)$ -null tree and let  $\bar{\varepsilon} = (\varepsilon_t : t \in \mathcal{T}) \subset (0, 1)$ . Then there is a full subtree  $(z_t : t \in \mathcal{T})$  of  $(x_t : t \in \mathcal{T})$  and a block tree  $(\tilde{z}_t : t \in \mathcal{T}) \subset S_X \cap c_{00}(E_j)$  with respect to  $(E_j)$  so that  $\|z_t - \tilde{z}_t\| < \varepsilon_t$  for all  $t \in \mathcal{T}$ . We say in that case that  $(\tilde{z}_t : t \in \mathcal{T})$  is a  $\bar{\varepsilon}$ -perturbation of  $(z_t : t \in \mathcal{T})$ .*

*Moreover, let  $\mathcal{T}$  be linearly ordered into  $t_0, t_1, t_2, \dots$  consistent with the partial order  $\succ$ , i.e., if  $m < n$ , then  $t_n$  and  $t_m$  are either incomparable with respect to  $\prec$  or  $t_m \prec t_n$ . Then  $(\tilde{z}_t : t \in \mathcal{T})$  can be chosen so that  $(\tilde{z}_{t_n})$  is a block sequence with respect to  $(E_j)$ .*

*Proof of Proposition 6.1.* Write  $\varepsilon_n = \varepsilon_{t_n}$ , and assume w.l.o.g. that  $\varepsilon_n < \frac{1}{2}$  for  $n \in \mathbb{N}$ . Choose  $\tilde{z}_\emptyset \in S_X \cap c_{00}(E_j)$  so that  $\|\tilde{z}_\emptyset - x_\emptyset\| < \varepsilon_0$ . Since the node  $(x_{\{n\}} : n \in \mathbb{N})$  is  $\sigma(X, Y)$ -null, and thus  $(P_{[1, \max \text{supp}_E(\tilde{z}_\emptyset)]}^E(x_{\{n\}}) : n \in \mathbb{N})$  is norm-null, we can choose  $k_1$  large enough, so that

$$\|P_{[1, \max \text{supp}_E(\tilde{z}_\emptyset)]}^E(x_{\{k_1\}})\| < \varepsilon_1/5$$

and choose  $s_1 = \{k_1\}$  (as element of  $\mathcal{T}$ ) and

$$z'_{\{1\}} = \frac{P_{(N, \infty)}^E(x_{\{k_1\}})}{\|P_{(N, \infty)}^E(x_{\{k_1\}})\|} \text{ and } z_{\{1\}} = x_{\{k_1\}},$$

where  $N = \max \text{supp}_E(\tilde{z}_\emptyset)$ . It follows that  $\|z'_{\{1\}} - z_{\{1\}}\| < \varepsilon_1$ . Indeed,

$$\left\| x_{\{k_1\}} - \frac{P_{(N, \infty)}^E(x_{\{k_1\}})}{\|P_{(N, \infty)}^E(x_{\{k_1\}})\|} \right\| \leq \|x_{\{k_1\}} - P_{(N, \infty)}^E(x_{\{k_1\}})\| + \|\|P_{(N, \infty)}^E(x_{\{k_1\}})\| - 1\| < \varepsilon_1.$$

Then we can perturb  $z'_{\{1\}}$  to an element  $\tilde{z}_{\{1\}}$  in  $S_X \cap c_{00}(E_j)$ , with  $\min \text{supp}(\tilde{z}_{\{1\}}) \geq N$  still satisfying  $\|\tilde{z}_{\{1\}} - z_{\{1\}}\| < \varepsilon_1$ .

Now assume that we have found  $s_0, s_1, s_2, \dots, s_{k-1} \in \mathcal{T}$  and a block sequence  $(\tilde{z}_{t_0}, \tilde{z}_{t_1}, \dots, \tilde{z}_{t_{k-1}})$  so that the set  $\mathcal{S}_{k-1} = \{s_0, s_1, s_2, \dots, s_{k-1}\}$  is close under taking restrictions, the map

$$\{t_0, t_1, t_2, \dots, t_{k-1}\} \rightarrow \{s_0, s_1, s_2, \dots, s_{k-1}\}, \quad t_j \mapsto s_j$$

is an order isomorphism, and  $\|\tilde{z}_{t_j} - x_{s_j}\| < \varepsilon_j$ , for  $j = 0, 1, \dots, k-1$ .

The element  $t_k$  has then a direct predecessor  $t_j$ ,  $j < k$ , with respect to  $\prec$  (not necessarily  $t_{k-1}$ ). Since the node  $(x_{s_j \cup \{n\}} : n > \max(s_j))$  is  $\sigma(X, Y)$ -null we can find a large enough  $n$

so that  $\|P_{[1,N]}^E(x_{s_j \cup \{n\}})\| < \varepsilon_k/5$  where  $N = \max \text{supp}_E(x_{s_{k-1}})$ . Then we let  $s_k = s_j \cup \{n\}$  and find as before  $\tilde{z}_{t_k} \in S_X \cap c_{00}(E_j)$ , so that  $\|\tilde{z}_{t_k} - x_{s_j \cup \{n\}}\| < \varepsilon_k$ , and note that the set  $\mathcal{S}_k = \{s_0, s_1, s_2, \dots, s_k\}$  is close under taking restrictions, and the map

$$\{t_j : j \leq k\} \rightarrow \{s_0, s_1, s_2, \dots, s_k\}, \quad t_j \mapsto s_j$$

is an order isomorphism.

This finishes the recursive construction, and we observe that  $(x_s : s \in \mathcal{S})$  with  $S = \bigcup_{k \in \mathbb{N}} \mathcal{S}_k$  is a full subtree and  $(\tilde{z}_t : t \in \mathcal{T})$  is an  $\bar{\varepsilon}$ -perturbation of that subtree, and, moreover, the sequence  $(\tilde{z}_{t_n} : n \in \mathbb{N}_0)$  is a block sequence in  $S_X$ .  $\square$

We now give a formal description of what it means that Player II has a winning strategy.

*Remark 6.2.* Assume  $\mathcal{A} \subset \mathcal{B}_\omega$ . Then Player II has a winning strategy in the  $\mathcal{A}$ -game if and only if there is a block tree  $(x_t : t \in \mathcal{T})$  so that no branch is in  $\mathcal{A}$ .

Indeed, for  $l \in \mathbb{N}$  and  $k_1 < k_2 < \dots, k_l$  we define  $x_{\{k_1, k_2, \dots, k_l\}}$ , to be the  $k$ -th choice of Player II following a winning strategy, assuming Player I has chosen so far  $k_1 < k_2 < k_3 < \dots < k_l$ . This defines a tree  $(x_t : t \in \mathcal{T})$  in  $S_X \cap (\oplus E_j)$ , which has the property that no branch of  $(x_t : t \in \mathcal{T})$  is in  $\mathcal{A}$ . Since  $\min \text{supp}_E(x_{\{k_1, k_2, \dots, k_l\}}) \geq k_l$ , we can pass to a full subtree of  $(x_t : t \in \mathcal{T})$  for which all nodes are block sequences.

Conversely, if there is a block tree  $(x_t : t \in \mathcal{T})$  so that no branch is in  $\mathcal{A}$  we can first assume, after passing to a full subtree, that  $\min \text{supp}_E(x_{\{k_1, k_2, \dots, k_l\}}) \geq k_l$  for all  $(k_1, \dots, k_l) \in \mathcal{T}$ . Player II can now use this tree as her strategy: If Player I has chosen  $k_1 < k_2 < \dots < k_l$  so far, Player II answers with  $x_{\{k_1, \dots, k_l\}}$ . The result of the game is therefore a branch of  $(x_t : t \in \mathcal{T})$ , which by assumption does not lie in  $\mathcal{A}$ .

**Proposition 6.3.** Assume  $\mathcal{A} \subset \mathcal{B}_\omega$  is closed and assume that  $(x_t : t \in \mathcal{T})$  is a winning strategy for Player II as in Remark 6.2. Then there exists a well founded and infinitely branching subtree  $(x_s : s \in \mathcal{S})$ , so that for every maximal  $s \in \mathcal{S}$

$$\{\bar{z} : \bar{z} \in \mathcal{B}_\omega, \bar{z} \succ \bar{x}_s\} \cap \mathcal{A} = \emptyset$$

Here we mean, as in Proposition 4.4, by  $\bar{x}_t$  for  $t = \{t_1, t_2, \dots, t_l\} \in \mathcal{T}$ , the finite sequence

$$\bar{x}_t = (x_{\{t_1\}}, x_{\{t_1, t_2\}}, x_{\{t_1, t_2, t_3\}}, \dots, x_{\{t_1, t_2, \dots, t_l\}})$$

*Remark 6.4.* Proposition 6.3 means that if  $\mathcal{A}$  is closed and Player II has a winning strategy, the outcome of the game is determined after finitely many (but possibly at the beginning of the game still undetermined) steps.

*Proof of Proposition 6.3.* Define

$$\mathcal{S}' = \{s \in \mathcal{T} : \bar{z} \in \mathcal{B}_\omega, \bar{z} \succ \bar{x}_s\} \cap \mathcal{A} \neq \emptyset \cup \{\emptyset\},$$

and note that  $\mathcal{S}'$  is closed under taking restrictions. Secondly, it is also well founded. Indeed, otherwise there would be an increasing sequence  $(k_j)$  in  $\mathbb{N}$  so that  $t_l = \{k_1, k_2, \dots, k_l\} \in \mathcal{S}'$ , for each  $l \in \mathbb{N}$ . But this would mean that for each  $l$  there is a block sequence  $z^{(l)} \in \mathcal{B}_\omega$  so that  $(\bar{x}_{t_l}, \bar{z}^{(l)}) \in \mathcal{A}$ . Since  $\mathcal{A}$  is closed this implies that the infinite sequence  $(x_{t_l} : l \in \mathbb{N})$  is in  $\mathcal{A}$ . Since  $(x_{t_l} : l \in \mathbb{N})$  is a branch of  $(x_t : t \in \mathcal{T})$  this contradicts the assumption we made for  $(x_t : t \in \mathcal{T})$ . Now define

$$\mathcal{S} = \{(s, n) : s \in \mathcal{S}', n \in \mathbb{N} \text{ with } n > \max(s)\}.$$

Then  $\mathcal{S}$  is also well founded, and no maximal element  $s$  of  $\mathcal{S}$  is in  $\mathcal{S}'$  and thus for every maximal element  $s$  in  $\mathcal{S}$  we have  $\{(\bar{x}_s, \bar{z}) : \bar{z} \in \mathcal{B}_\omega\} \cap \mathcal{A} = \emptyset$ . Moreover, every element which



is not maximal in  $\mathcal{S}$  must be in  $\mathcal{S}'$  and has therefore by definition of  $\mathcal{S}$  infinitely many successors.  $\square$

The following result was shown by Martin [21] for more general games. In the case that  $\mathcal{A} \subset \mathcal{B}_\omega$  is closed it has an easy proof (see also [21]).

**Theorem 6.5.** [21, Theorem] *If  $\mathcal{A} \subset \mathcal{B}_\omega$  is Borel then the  $\mathcal{A}$ -game is determined, meaning that either Player I or Player II has a winning strategy.*

*Remark 6.6.* From [21] it actually follows that it is enough that  $\mathcal{A}$  is Borel with respect to the product topology of the discrete topology on  $S_X \cap c_{00}(E_j)$ , to imply that the  $\mathcal{A}$ -game is determined.

**Definition 6.7.** We say that  $\mathcal{A} \subset \mathcal{B}_\omega$  is *closed under taking tails* if for every  $(x_j : j \in \mathbb{N}) \in \mathcal{A}$  and any  $n \in \mathbb{N}$  it follows that  $(x_{j+n} : j \in \mathbb{N})$  is in  $\mathcal{A}$ .

**Proposition 6.8.** *Assume that  $(\mathcal{A}_m : m \in \mathbb{N})$  is an increasing family of closed subsets of  $\mathcal{B}_\omega$  which are closed under taking tails and let  $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$ . If Player I has a winning strategy for the  $\mathcal{A}$ -game then there is an  $m \in \mathbb{N}$ , so that she also has a winning strategy for the  $\mathcal{A}_m$ -game.*

*Remark 6.9.* Let us first present an intuitive argument for the claim in Proposition 6.8. If Player II has a strategy for each  $\mathcal{A}_m$  game, for all  $m \in \mathbb{N}$ , she can use the following winning strategy for the  $\bigcup_{m \in \mathbb{N}} \mathcal{A}_m$ -game: First she follows her strategy for  $\mathcal{A}_1$  and choses  $x_1, x_2, x_3, \dots$ . Since  $\mathcal{A}_1$  is closed, she will, after say  $l_1$  moves, be in the situation that for all  $\bar{z} \in \mathcal{B}_\omega$ , with  $\bar{z} \succ (x_1, x_2, \dots, x_{l_1})$ , it follows that  $\bar{z} \notin \mathcal{A}$ . Then she switches to the strategy for  $\mathcal{A}_2$ , and after Player I chooses  $k_{l_1+1}$  she chooses the element  $x_{l_1+1}$  of  $S_X \cap c_{00}(E_j)$  which she would have chosen, if  $k_{l_1+1}$  had been the first step of Player I in the  $\mathcal{A}_2$ -game. She follows her strategy choosing  $x_{l_1+2}, x_{l_1+3}, \dots$  until, after some  $l_2$  steps, with  $l_2 > l_1$ , she will again be in the situation that for all  $\bar{z} \in \mathcal{B}_\omega$ , with  $\bar{z} \succ (x_{l_1+1}, x_{l_1+2}, \dots, x_{l_2})$ , it follows that  $\bar{z} \notin \mathcal{A}$ . She continues that way and finally produces a sequences  $(x_j) \in \mathcal{B}_\omega$  and  $(l_j) \subset \mathbb{N}$  so that  $(x_{l_m+j} : j \in \mathbb{N}) \notin \mathcal{A}_m$  for all  $m \in \mathbb{N}$ . Since  $\mathcal{A}$  is closed under taking tails, it follows that the whole sequence  $(x_j)$  is not in  $\mathcal{A}$  and, thus, that Player II has won.

Since by Theorem 6.5 the games  $\mathcal{A}_m$ ,  $m \in \mathbb{N}$ , and  $\mathcal{A}$  are determined, we deduce therefore that, if Player I has a winning strategy for the  $\mathcal{A}$  game, and thus player II was not a winning strategy for that game, it follows that there is an  $m \in \mathbb{N}$  so that player II has no winning strategy for the  $\mathcal{A}_m$ -game, and thus player I has a winning strategy for that game.

*Proof of Proposition 6.8.* Since by Theorem 6.5 the  $\mathcal{A}$ -game and the  $\mathcal{A}_m$ -games,  $m \in \mathbb{N}$ , are determined, we need to show that Player II has a strategy for the  $\mathcal{A}$ -game, assuming that she has a strategy for each  $\mathcal{A}_m$ -game. By Proposition 6.3 there is for each  $m \in \mathbb{N}$  a well founded and infinitely branching tree  $(x_s^{(m)} : s \in \mathcal{S}_m) \subset S_X \cap c_{00}(\oplus E_j)$ ,  $\mathcal{S}_m \subset \mathcal{T}$ , so that  $\{\bar{z} : \bar{z} \in \mathcal{B}_\omega, \bar{z} \succ \bar{x}_s^{(m)}\} \cap \mathcal{A}_m = \emptyset$ , for each maximal  $s \in \mathcal{S}_m$ . After relabeling we can assume that for each non maximal  $s = \{k_1, \dots, k_l\}$  in  $\mathcal{S}_m$  it follows that  $\{k_1, k_2, \dots, k_l, k\} \in \mathcal{S}_m$  for all  $k > k_l$ . We define a full tree  $(x_t; t \in \mathcal{T})$  as follows: If  $t = \emptyset$  we put  $x_\emptyset = x_\emptyset^{(1)}$  (this choice is irrelevant) For any other  $t = \{k_1, k_2, \dots, k_l\} \in \mathcal{T}$ ,  $l \geq 1$  we proceed as follows. We choose  $m \in \mathbb{N}$  and  $0 = l_0 < l_1 < l_2 < \dots < l_{m-1} < l_m = l$ , so that for all  $1 \leq j < m$ ,  $\{k_{l_{j-1}+1}, k_{l_{j-1}+2}, \dots, k_{l_j}\}$  is a maximal element of  $\mathcal{S}_j$  and  $\{k_{l_{m-1}+1}, k_{l_{m-1}+2}, \dots, k_{l_m}\}$  is a (not necessary maximal) element of  $\mathcal{S}_m$ . Then we define for that  $t$

$$x_t = x_{\{k_{l_{m-1}+1}, k_{l_{m-1}+2}, \dots, k_{l_m}\}}^{(m)}.$$

It follows that each branch  $(z_j)$  of  $(x_t : t \in \mathcal{T})$  (i.e.,  $z_j = x_{k_1, k_2, \dots, k_j}$  for some increasing sequence  $(k_j) \subset \mathbb{N}$ ) can be subdivided into finite sequences  $(z_j : l_{m-1} < j \leq l_m)$ , for  $m \in \mathbb{N}$ , so that for all  $\bar{z} \succ (z_j : l_{m-1} < j \leq l_m)$  we have  $\bar{z} \notin \mathcal{A}_m$ . In particular,  $(z_j : k_{m-1} < j) \notin \mathcal{A}_m$ , for all  $m \in \mathbb{N}$ , and since  $\mathcal{A}_m$  is closed under taking tails, it follows that  $(z_j : j \in \mathbb{N}) \notin \mathcal{A}$ . Thus  $(x_t : t \in \mathcal{T})$  is a winning strategy for Player II.  $\square$

Let us list some examples of sets  $\mathcal{A} \subset \mathcal{B}_\omega$  which are of interest (see [9, 16, 17, 22, 23, 24])

**Examples 6.10.** The following sets in (a), (b) (c)  $\mathcal{A} \subset \mathcal{B}_\omega$  are hereditary under taking tails and closed. The example in (d) is Borel.

a) For  $C \geq 1$  let

$$\mathcal{A} = \{(x_j) \in \mathcal{B}_\omega : (x_j) \text{ is } C\text{-unconditional}\}$$

b) For  $C \geq 1$  and  $1 \leq p \leq \infty$

$$\mathcal{A} = \{(x_j) \in \mathcal{B}_\omega : (x_j) \text{ is } C\text{-equivalent to the } \ell_p\text{-unit vector basis}\} \text{ or}$$

$$\mathcal{A} = \{(x_j) \in \mathcal{B}_\omega : (x_j) \text{ } C\text{-dominates the } \ell_p\text{-unit vector basis}\} \text{ or}$$

$$\mathcal{A} = \{(x_j) \in \mathcal{B}_\omega : (x_j) \text{ is } C\text{-dominated by the } \ell_p\text{-unit vector basis}\}$$

We could replace in the examples of (b) the  $\ell_p$  unit vector basis by any other basic sequence  $(v_j)$ . But in the case that  $(v_j)$  is not sub symmetric (if for example  $(v_j)$  is the unit vector basis of a Tsirelson space) the following choice is more meaningful (cf. [9, 24]).

c) Let  $(v_j)$  be a normalized basic sequence and  $C \geq 1$

$$\mathcal{A} = \left\{ (x_j) \in \mathcal{B}_\omega : \begin{array}{l} (x_j) \text{ is } C\text{-equivalent to } (v_{m_j}), \text{ where for } j \in \mathbb{N} \\ m_j \in [\max \text{supp}_E(x_j), \max \text{supp}_E(x_j)] \end{array} \right\} \text{ or}$$

$$\mathcal{A} = \left\{ (x_j) \in \mathcal{B}_\omega : \begin{array}{l} (x_j) \text{ } C\text{-dominates } (v_{m_j}), \text{ where for } j \in \mathbb{N} \\ m_j \in [\max \text{supp}_E(x_j), \max \text{supp}_E(x_j)] \end{array} \right\} \text{ or}$$

$$\mathcal{A} = \left\{ (x_j) \in \mathcal{B}_\omega : \begin{array}{l} (x_j) \text{ is } C\text{-dominated by } (v_{m_j}), \text{ where for } j \in \mathbb{N} \\ m_j \in [\max \text{supp}_E(x_j), \max \text{supp}_E(x_j)] \end{array} \right\}.$$

d) For the next example we assume that  $\mathcal{F} \subset [\mathbb{N}]^{<\omega}$  is hereditary, spreading and compact,  $C \geq 1$  and  $(v_j)$  is a normalized and subsymmetric basic sequence

$$\mathcal{A} = \{(x_j) \in \mathcal{B}_\omega : \{A \in [\mathbb{N}]^{<\omega} : (x_j : j \in A) \text{ is } C\text{-equivalent to } (v_j : j \in A)\} \in \mathcal{F}\}$$

$$\mathcal{A} = \{(x_j) \in \mathcal{B}_\omega : \{A \in [\mathbb{N}]^{<\omega} : (x_j : j \in A) \text{ } C\text{-dominates } (v_j : j \in A)\} \in \mathcal{F}\}$$

$$\mathcal{A} = \{(x_j) \in \mathcal{B}_\omega : \{A \in [\mathbb{N}]^{<\omega} : (x_j : j \in A) \text{ is } C\text{-dominated by } (v_j : j \in A)\} \in \mathcal{F}\}$$

Note that the first set in (d) can be written as

$$\mathcal{A} = \bigcap_{B \in [\mathbb{N}]^{<\omega} \setminus \mathcal{F}} \{(x_j) \in \mathcal{B}_\omega : (x_j : j \in B) \text{ is not } C\text{-equivalent to } (v_j : j \in B)\}.$$

This implies easily that  $\mathcal{A}$  is Borel. A similar argument works for the two other sets.

We are now ready to state the main result of this section

**Theorem 6.11.** *Let  $\mathcal{A} \subset \mathcal{B}_\omega$ . The following are equivalent.*

- For all decreasing sequences  $\bar{\varepsilon} = (\varepsilon_n) \subset (0, 1)$  Player I has a winning strategy for the  $\overline{\mathcal{A}_{\bar{\varepsilon}}}$ -game.
- For all decreasing sequences  $\bar{\varepsilon} = (\varepsilon_n) \subset (0, 1)$  every block tree  $(x_t : t \in \mathcal{T}) \subset S_X$  has a branch which lies in  $\overline{\mathcal{A}_{\bar{\varepsilon}}}$ .

- c) For all decreasing sequences  $\bar{\varepsilon} = (\varepsilon_n) \subset (0, 1)$  there is an increasing sequence  $(m_j) \subset \mathbb{N}$  so that for the blocking  $(H_j)$ , with  $H_j = \text{span}(E_i : m_{k-1} < i \leq m_k)$  (with  $m_0 = 0$ ), the following holds: Every normalized skipped block sequence  $(z_i) \subset S_X$  with respect to  $(H_j)_{j \geq 2}$  lies in  $\overline{\mathcal{A}_{\bar{\varepsilon}}}$ .

and letting  $Y$  be the closed linear span of  $(F_j)$ , then above conditions are equivalent with

- d) For all decreasing sequences  $\bar{\varepsilon} = (\varepsilon_n) \subset (0, 1)$  every  $\sigma(X, Y)$  null tree  $(x_t : t \in \mathcal{T}) \subset S_X$  has a branch which lies in  $\overline{\mathcal{A}_{\bar{\varepsilon}}}$ .

*Proof of Theorem 6.11.* The equivalences of (a), (b) and, (d), follow from Proposition 6.1, and Remark 6.2. It is also clear that (c) implies (a). Indeed, assuming (c), Player I has the following easy strategy: She chooses for given  $\varepsilon = (\varepsilon_n) \subset (0, 1)$  the sequence  $(m_j)$  as in (d). Her first move will be  $n_1 = m_1$ , and after the  $k$ -th step, in which Player II has chosen  $x_k$ , Player I choose  $n_{k+1} = m_{N+1}$ , where  $N = \max \text{supp}_H(x_k)$ . Therefore she forces Player II to pick a skipped block sequence with respect to  $(H_j)_{j \geq 2}$  which lies in  $\overline{\mathcal{A}_{\bar{\varepsilon}}}$ .

Now assume (a), our goal is to prove (c). Let  $\bar{\varepsilon} = (\varepsilon_j) \subset (0, 1)$  be given. We can assume that  $(\varepsilon_j)$  decreases. For  $n \in \mathbb{N}$  put  $\bar{\varepsilon}(n) = (\varepsilon_j(n) : j \in \mathbb{N}) = (\varepsilon_j(1 - 2^{-n}) : j \in \mathbb{N})$

We claim that we can recursively choose  $m_1 < m_2 < m_3 < \dots$  satisfying the following two properties (letting  $H_j = \text{span}(E_i : m_{j-1} < i \leq m_j)$ ,  $j = 1, 2, \dots, n$ ):

- (36) For every skipped block sequence  $(x_i)_{i=1}^l$  in  $S_X \cap \text{span}(H_j : 2 \leq j \leq n-1)$  (with respect to  $(H_j)$ ) and every  $x \in S_X \cap \text{span}(E_j : j > m_n)$  Player 1 has a winning strategy for the  $\overline{\mathcal{A}_{\bar{\varepsilon}(n)}}(x_1, \dots, x_l, x)$ -game.
- (37) For every skipped block  $(x_i)_{i=1}^l$  in  $S_X \cap \text{span}(H_j : 2 \leq j \leq n)$  (with respect to  $(H_j)$ ) Player 1 has a winning strategy for the  $\overline{\mathcal{A}_{\bar{\varepsilon}(n)}}(x_1, \dots, x_l)$ -game.

Since by assumption (d) Player 1 has a winning strategy for  $\overline{\mathcal{A}_{\bar{\varepsilon}(1)}}$  there is an  $m_1$ , so that for all  $x \in S_X \cap \text{span}(E_j : j \geq m_1)$  Player 1 has a winning strategy for the  $\overline{\mathcal{A}_{\bar{\varepsilon}(1)}}(x)$ -game. Note that for  $n = 1$ ,  $\emptyset$  is the only skipped block in  $\text{span}(H_j : 2 \leq j \leq 0)$ . Thus, in that case (36) simply says that for any  $x \in S_X \cap \text{span}(E_j : j > m_1)$  Player 1 has a winning strategy for the  $\overline{\mathcal{A}_{\bar{\varepsilon}(1)}}(x)$ -game, which follows from our choice of  $m_1$  and (37) means that Player 1 has a winning strategy for the  $\overline{\mathcal{A}_{\bar{\varepsilon}(1)}}$ -game, which follows from our assumption (d).

Now assume that  $m_1 < m_2 < \dots < m_n$  have been chosen so that conditions (36) and (37) hold. We first choose a *dense enough* finite set  $B$  of skipped block sequences with respect to  $(H_j : j = 2, 3, \dots, n)$ , more precisely,  $B$  includes the empty block, and for any skipped block sequence  $(x_j)_{j=1}^l$  in  $S_X$  with respect to  $(H_j : j = 2, \dots, n)$ , there is a sequence  $b = (\tilde{x}_j)_{j=1}^l \in B$  of the same length  $l$ , so that  $\text{supp}_E(x_j) = \text{supp}_E(\tilde{x}_j)$ , for  $j = 1, 2, \dots, l$ , and so that  $\|x_j - \tilde{x}_j\| < \varepsilon_{n+2}2^{-n-2}$ . Then we choose, using (37), to each  $b \in B$  a natural number  $k(b) > m_n$ , so that  $k(b)$  could be the first move of a winning strategy for Player 1 in the  $\overline{\mathcal{A}_{\bar{\varepsilon}(n)}}(b)$ -game. We let  $m_{n+1} = \max_{b \in B} k(b)$  and have to verify (36) and (37) for  $n+1$ .

To verify (36) for  $n+1$  let  $(x_j)_{j=1}^l$  be a skipped block in  $S_X \cap \text{span}(H_j : 2 \leq j \leq n)$  with respect to  $(H_j)_{j=2}^n$ . We first choose  $(\tilde{x}_j)_{j=1}^l \in B$ , so that  $\text{supp}_E(x_j) = \text{supp}_E(\tilde{x}_j)$ , and so that  $\|x_j - \tilde{x}_j\| < \varepsilon_{n+2}2^{-n-2}$ , for  $j = 1, 2, \dots, l$ . Note that  $\overline{\mathcal{A}_{\bar{\varepsilon}(n)}}(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_l) \subset \overline{\mathcal{A}_{\bar{\varepsilon}(n+1)}}(x_1, x_2, \dots, x_l)$ . Indeed,

$$\begin{aligned} \overline{\mathcal{A}_{\bar{\varepsilon}(n)}}(\tilde{x}_1, \dots, \tilde{x}_l) &= \overline{\{(z_j) \subset S_X : (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_l, z_1, z_2, \dots) \in \mathcal{A}_{\bar{\varepsilon}(n)}\}} \\ &\subset \overline{\{(z_j) \subset S_X : (x_1, x_2, \dots, x_l, z_1, z_2, \dots) \in \mathcal{A}_{\bar{\varepsilon}(n+1)}\}} = \overline{\mathcal{A}_{\bar{\varepsilon}(n+1)}}(x_1, \dots, x_l). \end{aligned}$$

The choice of  $m_{n+1}$  therefore yields that condition (36) is satisfied for  $n+1$ . Condition (37) for  $n+1$  follows now from condition (36) for  $n$  if we note that for any normalized sequence  $(x_i)_{i=1}^l \subset S_X \cap \text{span}(H_j : 2 \leq j \leq n+1)$ , which is a skipped block with respect to  $(H_j)_{j=2}^{n+1}$  the sequence  $(x_j)_{j=1}^{l-1}$  must be a skipped block sequence in  $\text{span}(H_j : 2 \leq j \leq n-1)$ . This finishes the inductive choice of  $(m_j)$  and  $(H_j)$ .

Now assume that  $(x_i)$  is a skipped block sequence with respect to  $(H_j)$  in  $S_X$ . Then for every initial segment  $(x_j)_{j=1}^l$  Player 1 has a winning strategy for  $\overline{\mathcal{A}_\varepsilon}(x_1, \dots, x_l)$  in particular this means that  $\mathcal{A}_\varepsilon(x_1, \dots, x_l)$  cannot be empty. Thus, since  $\overline{\mathcal{A}_\varepsilon}$  is closed it follows that  $(x_j) \in \overline{\mathcal{A}_\varepsilon}$ .  $\square$

From Theorem 6.11 we deduce the missing part of the Main Theorem, namely the verification that, under the appropriate assumption, the FDD  $(Z_i)$  of  $Z$  is unconditional.

According to [17] we say that  $X$  has the  $w^*$ -Unconditional Tree Property ( $w^*$ -UTP) if every  $w^*$ -null tree in  $X^*$  has a branch which is unconditional.

**Corollary 6.12.** [17] *Assume that  $X$  has the  $w^*$ -UTP and that  $X^*$  is separable. Then there is an FMD  $(E_j)$  with biorthogonal sequence  $(F_j)$  so that the space  $\text{FDD}(Z_j)$  of the space  $Z$ , as constructed in Section 3 is unconditional.*

*Proof.* Let  $(E'_j)$  be any shrinking FMD of  $X$  and  $(F'_j)$  its biorthogonal sequence and define for  $C \geq 1$ .

$$\mathcal{A}_C = \{(x_j^*) \in \mathcal{B}_\omega(X^*, F') : (x_j^*) \text{ is } C\text{-unconditional}\} \text{ and}$$

$$\mathcal{A} = \bigcup_{m \in \mathbb{N}} \mathcal{A}_m = \{(x_j^*) \in \mathcal{B}_\omega(X^*, F') : (x_j^*) \text{ is unconditional}\}.$$

As noted in Examples 6.10  $\mathcal{A}_C$  is closed. We also note that for any summable and decreasing sequence  $\bar{\varepsilon} = (\varepsilon_j) \subset (0, 1)$  we have  $\mathcal{A} = \mathcal{A}_{\bar{\varepsilon}}$ , and for  $C \geq 1$  there is a  $C' = C'(\varepsilon)$  so that  $\mathcal{A}_C \subset [\mathcal{A}_C]_{\bar{\varepsilon}} \subset \mathcal{A}_{C'}$ . Using the equivalence (a)  $\iff$  (d) in Theorem 6.11 and Proposition (6.8) we deduce that there is a  $C \geq 1$  so that Player I has a winning strategy for the  $\mathcal{A}_C$ -game. But this implies, maybe after increasing  $C$  slightly and using the equivalence (a)  $\iff$  (c) in Theorem 6.11 that we can block  $(F'_j)$  into an MFD  $(F_n)$  so that every skipped block in  $S_{X^*} \cap \text{span}(F : j \geq 2)$  with respect to  $(F_j)$  is  $C$ -unconditional. After possibly increasing  $C$  again and after possibly passing to further blocks, we can assume that every skipped block in  $S_{X^*} \cap \text{span}(F : j \geq 1)$  with respect to  $(F_j)$  is  $C$ -unconditional and that the conclusions of Lemma 2.3 are satisfied. Therefore our claim follows from Proposition 3.8.  $\square$

*Remark 6.13.* As proved in [16, Theorem 2.12] if  $X$  is reflexive the property of having the  $w^*$ -Unconditional Tree Property is equivalent with having the  $w$ -Unconditional Tree Property which means that every weakly null tree in  $S_X$  (not in  $X^*$ ) has a branch which is unconditional.

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